MATH 135 Exam Equation Sheet

PREP101 Your Competitive Edge

REVIEW

Linear quadratic equation:

- for $ax^2 + bx + c = 0$, variable

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Logarithms:

- $\log_b(x \cdot y) = \log_b(x) + \log_b(y)$
- $\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$
- $\log_b(x^n) = n \cdot \log_b(x)$

Exponent laws:

- $b^m \cdot b^n = b^{m+n}$
- $\left(\frac{b^m}{b^n}\right) = b^{m-n}$
- $(b^m)^n = b^{m \cdot n}$
- $b^1 = b$
- $b^0 = 1$

Trigonometry:

- $\sin^2 u + \cos^2 u = 1$
- $1 + \tan^2 u = \sec^2 u$
- $1 + \cot^2 u = \csc^2 u$
- $\sin(u \pm v) = \sin u \cos v \pm \cos u \sin v$
- $\cos(u \pm v) = \cos u \cos v \mp \sin u \sin v$
- $\tan(u \pm v) = \frac{\tan u \pm \tan v}{1 \mp \tan u \tan v}$

LIMITS AND CONTINUITY
Definition and Properties

Definition of the Limit:

- The function *f(x)* has limit *L*, as *x* approaches *a*, denoted $\lim_{x \to a} f(x) = L$ if given any $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ for all *x* satisfying $0 < |x - a| < \delta$.

Properties of limits:

- $\lim_{x \to a}[f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$
  $\lim_{x \to a}[f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$
- $\lim_{x \to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$
- $\lim_{x \to a}[f(x)]^n = [\lim_{x \to a} f(x)]^n$

LIMITS AND CONTINUITY
Techniques for Finding Limits

Steps to Find a Limit

- First try substitution, factoring
- If you're taking the limit as $x \to \infty$ of a quotient, and substitution yields an invalid answer, then try first to divide both the numerator and the denominator of the limit expression by its highest power of *x*
- If the limit expression contains ***absolute values***, then try breaking the limit up into *two one-sided limits*.
- If the limit expression is a quotient, and if factoring doesn't work, then try l'Hopital's Rule: $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

L'Hopital's rule:

- If the limit of the quotient of differentiable functions $f(x)$ and $g(x)$ are of types $\frac{0}{0}$ or $\frac{\infty}{\infty}$, and if $g'(a)$ is not 0, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

THE DERIVATIVE
First Principles, Rules and Derivatives to Remember

Differentiation by first principles:

- $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

Rules for differentiation:

- for functions *u*,*v* and for constants *c*, *n*:
  $\frac{dy}{dx}(cu^n) = cnu^{n-1}\frac{du}{dx}$
- $\frac{dy}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$
- $\frac{dy}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Chain rule:

- If $f(x) = (a \circ b)(x)$, then $f'(x) = a'(b(x)) \cdot b'(x)$

Power rule:

- $\frac{d}{dx}c^{u(x)} = c^{u(x)} \cdot \ln c \cdot \frac{du}{dx}$

Derivatives to remember:

- $\frac{d}{du}(\ln u) = \frac{1}{u} \cdot \frac{du}{dx}$
- $\frac{d}{dx}(e^u) = e^u \frac{du}{dx}$
- $\frac{d}{dx}(a^u) = a^u(\ln a)\frac{du}{dx}$
- $\frac{d}{dx}(\sin x) = \cos x$
- $\frac{d}{dx}(\cos x) = -\sin x$
- $\frac{d}{dx}(\tan x) = \sec^2 x$
- $\frac{d}{dx}(\cot x) = -\csc^2 x$
- $\frac{d}{dx}(\sec x) = \sec x \cdot \tan x$
- $\frac{d}{dx}(\csc x) = -\csc x \cdot \cot x$

Higher order derivatives:

- $f''(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right)$.

THE DERIVATIVE
Techniques of Differentiation

Implicit differentiation:

- to differentiate an implicit function
- to differentiate a function with an *x* in both the base and the exponent

Step 1. Take the derivative of both sides with respect to *x*. Use the Chain Rule on terms involving *y* (and note that the derivative of y with respect to *x* must be left as dy/dx.)
Step 2. Collect all terms involving dy/dx on one side of the equation.
Step 3. Solve for dy/dx.

Logarithmic differentiation:

- to differentiate a function with complicated exponent/a product of several functions, etc.

Step 1. Take the 'ln' of both sides (to find an expression of the form $\ln y = \ln[f(x)]$).
Step 2. Simplify $\ln(f(x))$ by using the properties of logarithms.
Step 3. Differentiate both sides with respect to *x* (thus: $\frac{1}{y} \cdot \frac{dy}{dx} = \frac{d}{dx}(\ln f(x))$).
Step 4. Solve for dy/dx.
Step 5. Express the answer in terms of *x* only (substitution *f(x)* for *y*).

APPLICATIONS OF THE DERIVATIVE

Critical points:

- The values of $x \in$ domain of *f(x)* such that $f'(x) = 0$ or $f'(x)$ is not defined.

First derivative test:

- If $f'(p) = 0$ and f' changes from negative to positive at *p*, then *f* has a relative minimum at *p*. If f' changes from positive to negative at *p*, then *f* has a relative maximum at *p*.

Second derivative test:

- If $f'(p) = 0$ and $f''(p) > 0$, then *f* has a relative minimum at *p*. If $f'(p) = 0$ and $f''(p) < 0$, then *f* has a relative maximum at *p*. If $f'(p) = 0$ and $f''(p) = 0$, then the test fails.

Concavity:

- If $f''(p) > 0$ on an interval, then $f(x)$ is concave up on that interval; if $f''(p) < 0$, then $f(x)$ is concave down on that interval. A point of inflection occurs when $f''(p)$ changes sign (and thus concavity).

Vertical asymptote:

- The line $x = a$ is a ***vertical asymptote*** for the graph of the function $f(x)$ if and only if $\lim_{x\to a+} f(x) = \pm\infty$ or $\lim_{x\to a-} f(x) = \pm\infty$.

Horizontal asymptote:

- The line $x = b$ is a ***horizontal asymptote*** for the graph of the function $f(x)$ if and only if $\lim_{x\to +\infty} f(x) = b$ or $\lim_{x\to -\infty} f(x) = b$

Curve sketching:

Step 1. Find the intercepts.
Step 2. Find all the asymptotes.
Step 3. Find critical points and the intervals of increase/decrease.
Step 4. Find inflection points and the intervals in which the function is concave up/down.

Optimization problems:

Step 1. Determine what we're trying to maximize/minimize and write an equation for this.
Step 2. Write a second equation from additional information given in the problem, isolate one of the two variables, and substitute this into the first equation.
Step 3. Take the derivative of this equation, set it to zero and solve for the remaining variable.
Step 4. Plug the value of this variable into the original equations to solve for the remaining variables.

INTEGRATION
Rules and Properties

Rules of integration:

- $\int dx = x + c$
- $\int x^n \cdot dx = \frac{x^{n+1}}{n+1} + C$
- $\int \frac{dx}{x} = \ln|x| + C$
- $\int \sin x dx = -\cos x + C$
- $\int \cos x dx = \sin x + C$
- $\int \tan x dx = \ln|\sec x| + C$

Properties of integration:

- $\int_a^b kf(x)\cdot dx = k\int_a^b f(x)\cdot dx$
- $\int_a^b [f(x) \pm g(x)]\cdot dx = \int_a^b f(x)\cdot dx \pm \int_a^b g(x)\cdot dx$
- $\int_a^b f(x)\cdot dx = \int_a^c f(x)\cdot dx + \int_c^b f(x)\cdot dx$

The fundamental theory of Calculus:

- For $F'(x) = f(x)$

$\int_a^b f(x)dx = F(b) - F(a)$

Odd and even functions:

- if $f(-x) = f(x)$, then $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$;
- if $f(-x) = -f(x)$, then $\int_{-a}^{a} f(x)dx = 0$.

INTEGRATION
Techniques of Integration

Integration by substitution

Step 1. Choose a substitution u such that the resulting integral, written in terms of u and du, will be easy to evaluate (easiest with practice).
Step 2. Differentiate u with respect to x to find $du = u'(x)dx$.
Step 3. Replace dx with $du/u'(x)$.
Step 4. Integrate with respect to u.
Step 5. Substitute back to the initial variable x.

Integration by parts

Step 1. Write the given integral $\int f(x)g(x)\cdot dx$.

Step 2. Introduce the intermediary functions u(x) and v(x):
$\begin{cases} u = f(x) \\ dv = g(x)dx \end{cases}$.

Step 3. Differentiate u and integrate dv to get:
$\begin{cases} du = f'(x) \\ v = \int g(x)dx \end{cases}$.

Step 4. Use the following formula:
$\int u(x)dv = u(x)v(x) - \int v(x)du$ and solve.

Partial fractions:

- to integrate rational functions of the form $f(x) = \frac{P(x)}{Q(x)}$

Step 1. If the degree (P) ≥ degree (Q), perform polynomial long-division. If it is not, proceed to step 2.
Step 2. Factor the denominator $Q(x)$ into irreducible polynomials: linear and irreducible quadratic polynomials.
Step 3. Find the partial fraction decomposition.
Step 4. Integrate the result of step 3.

Integration by Trigonometric substitution

- To be used when the integrand contains the expressions $\sqrt{a^2 \pm x^2}$ or $\sqrt{x^2 \pm a^2}$.
- For $\sqrt{a^2 - x^2}$ set $x = a\sin(t)$
- For $\sqrt{a^2 + x^2}$ set $x = a\tan(t)$,
- For $\sqrt{x^2 - a^2}$ set $x = a\sec(t)$

Improper Integration

- If the integral exists (and equals, or can be written as, a number), then it's **convergent**
- If the integral doesn't exist (is infinite), then it's **divergent**

INTEGRATION
Applications

Area:

- Area between functions $f(x)$ and $g(x)$ (whose intersection points are a and b, and where $f(x) > g(x)$ on (a,b)) is given by $\int_a^b [f(x) - g(x)]dx$.

Average Value:

- The Average value of function $f(x)$ on the interval $[a, b]$ (or (a, b)) is given by $\bar{f} = \frac{1}{b-a}\int_a^b f(x)dx$.

Arc Length:

- Arc Length of the function $f(x)$ from $x = a$ to $x = b$ is given by $\int_a^b \sqrt{1 + (f'(x))^2}\cdot dx$.

Surface Area:

- Surface area from a to b of the solid obtained by revolving $y = f(x)$ around the x-axis is given by $2\pi\int_a^b f(x)\cdot\sqrt{1+(f'(x))^2}\cdot dx$

Volume:

- If the cross section is perpendicular to the x-axis and its area is a function of x, say $A(x)$, then the volume is $V = \int_a^b A(x)dx$.
- If the cross section is perpendicular to the y-axis and its area is a function of y, say $A(y)$, then the volume is $V = \int_a^b A(y)dy$

Volumes of solids by revolution:

- The volume of the solid obtained by rotating the graph of $f(x)$ from $x=a$ to $x=b$ about the y-axis is given by $2\pi\int_a^b x\cdot f(x)\cdot dx$
- The volume of the solid obtained by rotating the region bounded between curves $f(x)$ and $g(x)$ between $x=a$ and $x=b$ (assume $g(x)>f(x)$ for $a<x<b$) about the line $y=c$ is given by $\pi\int_a^b\left[(c-g(x))^2-(c-f(x))^2\right]dx$.

Centroid:

- The moment about the *y*-axis is given by $M_y=\lim_{n\to\infty}\sum_{i=1}^n \rho\bar{x}_i f(\bar{x}_i)\Delta x=\rho\int_a^b xf(x)dx$
- the moment about the *x*-axis is given by $M_x=\lim_{n\to\infty}\sum_{i=1}^n \rho\cdot\frac{1}{2}\left[f(\bar{x}_i)\right]^2\Delta x=\rho\int_a^b\frac{1}{2}\left[f(x)\right]^2dx$
- The center of mass of the centroid is at the point $(\bar{x},\bar{y})$ given by
$\bar{x}=\frac{1}{A}\int_a^b xf(x)dx$
$\bar{y}=\frac{1}{A}\int_a^b\frac{1}{2}[f(x)]^2dx$

DIFFERENTIAL EQUATIONS

Separation of variables:

- for functions of the form $\frac{dy}{dx}=f(x)\cdot g(y)$

Step 1. place the terms involving the variable x on one side and the terms involving the variable y on the other side (and find $f(x)\cdot dx=\frac{1}{g(y)}\cdot dy$).

Step 2. Integrate both sides of the equation ($\int f(x)dx=\int\frac{1}{g(y)}dy$) and solve.

Integrating Factors:

- for function of the form $\frac{dy}{dx}+p(x)\cdot y=q(x)$

Step 1. Create the integrating factor as follows: $I(x)=e^{\int p(x)\cdot dx}$.

Step 2. Solve the differential equation as follows: $y=\frac{\int I(x)\cdot q(x)\cdot dx}{I(x)}$.

SEQUENCES AND SERIES
Definitions, Rules and Properties

Arithmetic:

- constant growth $a_{n+1}=a_n+d$ where *d* is a constant

Geometric:

- proportional growth $g_{n+1}=rg_n$ where *r* is a constant, arbitrary terms are given by $g_n=r^{n-1}\cdot g_1$.

Fibonacci:

- defined recursively with $a_1=1$, $a_2=1$, $a_n=a_{n-1}+a_{n-2}$ for $n\geq 3$

Algebra of sequences:

- $\{a_n\}\pm\{b\}=\{a_n\pm b_n\}$
- $\{a_n\}\cdot\{b_n\}=\{a_n\cdot b_n\}$
- $c\cdot\{a_n\}=\{c\cdot a_n\}$

Algebra of series:

- $\sum_{n=1}^N a_n\pm\sum_{n=1}^N b_n=\sum_{n=1}^N(a_n\pm b_n)$
- $\sum_{n=c}^N a_n=\sum_{n=1}^N a_n-\sum_{n=1}^{c-1}a_n$

Summation formulas:

- $\sum_{n=1}^N n=\frac{N\cdot(N+1)}{2}$
- $\sum_{n=1}^N n^2=\frac{N\cdot(N+1)\cdot(2N+1)}{6}$
- $\sum_{n=M}^N a\cdot r^{n-1}=\frac{a\cdot(r^{M-1}-r^N)}{1-r}$

SEQUENCES AND SERIES
Convergence Tests

P-series:

- A P-series $\sum_{n=1}^\infty\frac{1}{n^p}$ is convergent if $p>1$, and is divergent if $p\leq 1$.

Integral test:

- let $\sum_{i=1}^\infty a_i$ be a series such that there exists a function $a_n=f(n)$ where *f* is continuous, positive and decreasing.
- If $\int_1^\infty f(x)\cdot dx$ converges, then the series converges and if it diverges, then the series diverges.

Comparison test:

- for $\sum_{i=1}^\infty a_n$ and $\sum_{i=1}^\infty b_n$ such that $0<a_n\leq b_n$ for all n.
- If $\sum_{i=1}^\infty b_n$ is convergent, then $\sum_{i=1}^\infty a_n$ is convergent. If $\sum_{i=1}^\infty a_n$ is divergent then $\sum_{i=1}^\infty b_n$ is divergent.

Alternating series test:

- for $\{a_n\}$ a sequence of positive numbers such that $a_i\geq a_{i+1}$ for all *i* and $\lim_{n\to\infty}a_n=0$, then $\sum_{i=1}^\infty(-1)^{n+1}a_n$ converges.

Limit comparison test:

- if $\lim_{n\to\infty}\frac{a_n}{b_n}=L$ such that $L\neq 0$ and $L\neq\infty$, then $\sum_k^\infty a_n$ and $\sum_k^\infty b_n$ converge or diverge together.

Ratio test:

- Given $a_n>0$ and $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=L$.
- If $L<1$, then $\sum_{i=1}^\infty a_n$ converges and if $L>1$, then $\sum_{i=1}^\infty a_n$ diverges.
- If $L=1$, then the test fails.

Root test:

- Given $a_n>0$ and $\lim_{n\to\infty}(a_n)^{1/n}=L$.
- If $L<1$, then $\sum_{i=1}^\infty a_n$ converges and if $L>1$, then $\sum_{i=1}^\infty a_n$ diverges.
- If $L=1$, then the test fails.

Absolute and conditional convergence:

- A series of the form $\sum a_n$ ***converges absolutely*** if the series of absolute values (of the form $\sum|a_n|$) is convergent.
- If it converges absolutely, then the original series $\sum a_n$ will also converge.