Solutions to Mat133 Practice Test#2

PART A: MULTIPLE CHOICE

1. *[4 marks]*

Solution: E

$$\lim_{x\to 0}\frac{\sin 2x\cos 3x}{\cos 4x\sin 5x}=\lim_{x\to 0}\frac{2}{5}\cdot\frac{\sin 2x}{2x}\cdot\frac{5x}{\sin 5x}\cdot\frac{\cos 3x}{\cos 4x}=$$
$$=\frac{2}{5}\times 1\times 1\times 1=\frac{2}{5}.$$

2. *[4 marks]*

Solution: D

Note first that this limit is in $\frac{0}{0}$ form. Therefore, use L'Hopital's theorem:

We find $\lim_{x\to 2}\frac{3x^2+6x-10}{3x^2-6x+2}=\frac{14}{2}=7$

3. *[4 marks]*

Solution: A

As $x\to 2^-$, $\frac{1}{x-2}\to -\infty$. Thus, $\lim_{x\to 2^-} e^{\frac{1}{x-2}} = \lim_{t\to -\infty} e^t = 0$.

4. *[4 marks]*

Solution: B

$$\ln f(x)=\ln x^{\sin x}=\sin x\ln x$$

Differentiate both sides to get:

$$\frac{1}{f(x)}f'(x)=(\sin x)\left(\frac{1}{x}\right)+(\cos x)(\ln x)$$

$$\Rightarrow f'(x)=f(x)\left[(\sin x)\left(\frac{1}{x}\right)+(\cos x)(\ln x)\right]\Rightarrow f'\left(\frac{\pi}{2}\right)=\left(\frac{\pi}{2}\right)^1\left[\frac{1}{\pi/2}+0\right]=1.$$

5. *[4 marks]*

Solution: C

$$P'(x)=3k(x^{\frac{1}{3}}+4)^2\left(\frac{1}{3}x^{-2/3}\right)$$

$$P'(x)=k(x^{\frac{1}{3}}+4)^2(x^{-2/3})$$

$$72=k(8^{\frac{1}{3}}+4)^2(8^{\frac{-2}{3}})$$

$k = \frac{72}{36(1/4)} = 8$

6. *[4 marks]*

Solution: C

Let $y = \tan^{-1}\sqrt{x}$ and $u = \sqrt{x} = x^{1/2}$. Then $y = \tan^{-1} u$.

$\frac{du}{dx} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$ and $\frac{dy}{du} = \frac{1}{1+u^2}$.

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{1+u^2}\left(\frac{1}{2\sqrt{x}}\right) = \frac{1}{1+\left(\sqrt{x}\right)^2}\left(\frac{1}{2\sqrt{x}}\right) = \frac{1}{2\sqrt{x}(1+x)}.$$

7. *[4 marks]*

Solution: B

$$\lim_{h\to 0}\frac{\ln\left|(2+h)^2+3\right| - \ln 7}{h}$$

$$= \lim_{h\to 0}\frac{\ln\left|(2+h)^2+3\right| - \ln\left|2^2+3\right|}{h} = \frac{d}{dx}\ln\left(x^2+3\right)\Big|_{x=2} = \frac{1}{x^2+3}(2x)\Big|_{x=2} = \frac{4}{2^2+3} = \frac{4}{7}$$

8. *[4 marks]*

Solution: C

By differentiating implicitly with respect to t:

$$\frac{dy}{dt} = \frac{d}{dt}\left(\frac{1}{x^2+4}\right) = \frac{d}{dx}\left(\frac{1}{x^2+4}\right)\frac{dx}{dt} = \frac{-2x}{(x^2+4)^2}\frac{dx}{dt},$$

So when $x = 2$: $\Rightarrow \frac{dy}{dt} = \frac{-2\cdot 2}{(2^2+4)^2}\cdot 3 = -\frac{3}{16}$ units/sec.

9. *[4 marks]*

Solution: B

False.

Consider non-zero matrices $A = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$ and $B = \begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix}$.

Note that $AB = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$

10. *[4 marks]*

Solution: D

$F'(x) = f'(g(x))g'(x) \Rightarrow F'(3) = f'(g(3))g'(3) = f'(6)g'(3) = 7 \times 4 = 28.$

PART B. Written-Answer Questions

1.

a) *[8 marks]*

Solution:

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x^2 = 4$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} cx - 4 = 2c - 4.$$

In order that f be continuous at 2, we must have $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$.

Hence, 2c – 4 = 4 $\Rightarrow$ c = 4.

Solution 2: x2 = cx – 4. At x = 2, 4 = 2c – 4 $\Rightarrow$ c = 4.

b) Does f'(2) exist? Justify your answer. *[8 marks]*

$$\lim_{x\to 2^-} \frac{f(x)-f(2)}{x-2} = \lim_{x\to 2^-} \frac{x^2-4}{x-2} = \lim_{x\to 2^-} x+2 = 4$$

and

$$\lim_{x\to 2^+} \frac{f(x)-f(2)}{x-2} = \lim_{x\to 2^+} \frac{4x-4-4}{x-2} = \lim_{x\to 2^+} \frac{4(x-2)}{x-2} = 4$$

Thus, $\lim_{x\to 2} \frac{f(x)-f(2)}{x-2}$ exists and f is continuous at x = 2. Hence, f'(2) does exist.

Solution 2: $f'(x) = \begin{cases} 2x & \text{if } 0 < x < 2 \\ 4 & \text{if } x > 2 \end{cases}$

$$\begin{cases} \lim_{x\to 2^-} f'(x) = \lim_{x\to 2^-} 2x = 4 \\ \lim_{x\to 2^+} f'(x) = \lim_{x\to 2^+} 4 = 4 \end{cases} \Rightarrow f'(2) = 4$$

. Also, f is continuous at x = 2. Thus, f'(2) exists.

2. *[14 marks]*

a) **Solution:**

We're given that the amount of money in the account is $A(t) = 500e^{(0.04)t}$.

Then if $1000 = A(t_0) = 500e^{(0.04)t_0}$, $\Rightarrow t_0 = \dfrac{\ln 2}{0.04}$ years.

b) **Solution:**

We find $A'(t) = 20e^{(0.04)t}$, $\Rightarrow A'\left(\dfrac{\ln 2}{0.04}\right) = 20e^{(0.04)\frac{\ln 2}{0.04}} = 20e^{\ln 2} = 40$ dollars/year.

3. *[12 marks]*

a)

Solution:

$$\frac{dq}{dp} = -\frac{q}{p}E(p) = -\frac{1000}{2}(0.4) = -200.$$

b) Use a linear approximation (tangent line approximation) to estimate the demand if the price rises to \$2.15 per unit.

Solution:

The tangent line to $q(p)$ when $p = 2$ is defined by the equation:

$$q = q(2) + q'(2)(p-2) = 1000 - 200(p-2), \ \Rightarrow q(2.15) \approx -200(2.15-2) + 1000 = 970.$$

4. *[18 marks]*

(a)

$$f'(x) = 1 + \frac{1}{2}|x|^{-1/2} = 1 + \frac{1}{2\sqrt{|x|}} > 0 \text{ for all } x \in \Re \text{ and } x \neq 0.$$

Hence, f is increasing for all $x \in \Re$.

(b)

$f(x)$ doesn't have local maximum or local minimum since $f'(x) > 0$ for all $x \neq 0$.

(c)

$$f''(x) = \frac{-1}{4}|x|^{-3/2} = -\frac{1}{4|x|^{3/2}}.$$

$f''(x) < 0$ for $x \neq 0$. Hence, f is concave downward for all $x \in \Re$.

(d)