Solutions

1)

a) The characteristic equation is $r^2+r+2=0$. Using the quadratic formula, we find

$$r=\frac{-1\pm\sqrt{1^2-4\times1\times2}}{2\times1}=\frac{-1\pm\sqrt{1-8}}{2}=\frac{-1\pm\sqrt{-7}}{2}$$
$$=\frac{-1\pm\sqrt{7}i}{2}=\frac{-1}{2}\pm\frac{1}{2}\sqrt{7}i$$

Since in this case, these are complex conjugate roots, the general solution is

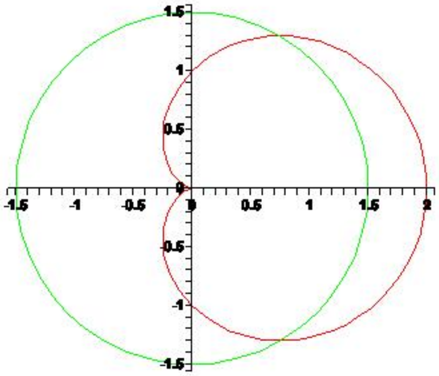
$$y(x)=Ae^{\frac{-1}{2}x}\cos(\frac{1}{2}\sqrt{7}x)+Be^{\frac{-1}{2}x}\sin(\frac{1}{2}\sqrt{7}x)$$

where A and B are constants.

b) We can write the equation as:

$$dy=\frac{dy}{dx}dx=y^2\cos x dx$$
$$\therefore\int\frac{dy}{y^2}=\int\cos x dx$$
$$\therefore\frac{-1}{y}=\sin x+c$$
$$\therefore y=\frac{-1}{\sin x+c}$$

2) We start by sketching the two curves:



To compute such area, we first need to find the point of intersection of these two curves, to do that, we solve the equation $1+\cos(\theta)=\frac{3}{2}$ to get: $\theta=-\frac{\pi}{3},\frac{\pi}{3}$.

Now, if we let $r_1 = 1+\cos\theta$ and $r_2 = \frac{3}{2}$, the area inside r₁ and outside r₂ is given by:

$$A = 2\cdot\left[\frac{1}{2}\int_0^{\pi/3} r_1^2\,d\theta - \frac{1}{2}\int_0^{\pi/3} r_2^2\,d\theta\right] = \left[\int_0^{\pi/3}(1+\cos(\theta))^2\,d\theta - \int_0^{\pi/3}\left(\frac{3}{2}\right)^2 d\theta\right] =$$

$$\left[\int_0^{\pi/3} d\theta + 2\cdot\int_0^{\pi/3}\cos(\theta)d\theta + \int_0^{\pi/3}\cos^2(\theta) - \frac{9}{4}\cdot\frac{\pi}{3}\right] = \left[\frac{\pi}{3} + 2\cdot\sin(\theta)\Big|_0^{\pi/3} + \int_0^{\pi/3}\left(\frac{1}{2}+\frac{1}{2}\cdot\cos(2\theta)\right) - \frac{3\pi}{4}\right] =$$

$$\left[-\frac{\pi}{4} + \sqrt{3} + \frac{1}{4}\sin(2\theta)\Big|_0^{\pi/3}\right] = \frac{9\sqrt{3}}{8} - \frac{\pi}{4}$$

3)

a) We start by finding the arclength element:

$$\frac{dx}{dt} = e^t(\cos(t)+\sin(t)) + e^t(-\sin(t)+\cos(t)) = 2e^t\cos(t)$$

$$\frac{dy}{dt} = e^t(\cos(t)-\sin(t)) + e^t(-\sin(t)-\cos(t)) = -2e^t\sin(t)$$

$$dl = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \sqrt{4e^{2t}\cos^2(t) + 4e^{2t}\sin^2(t)}dt = \sqrt{4e^{2t}[\cos^2(t)+\sin^2(t)]}dt$$

Therefore $L = \int_{-\ln(3)}^{\ln(3)} 2e^t\,dt = 2e^t\Big|_{-\ln(3)}^{\ln(3)} = 2e^{\ln(3)} - 2e^{-\ln(3)} = 2\cdot 3 - \frac{2}{3} = \frac{16}{3}$

b) The Binomial series for $(1+t)^{1/4} = \sum_{n=0}^{\infty}\binom{1/4}{n}t^n$, where

$$\binom{\frac{1}{4}}{n} = \frac{\frac{1}{4}\left(-\frac{3}{4}\right)\left(-\frac{7}{4}\right)\left(-\frac{11}{4}\right)\ldots\left(\frac{1}{4}-n+1\right)}{n!}.$$

Hence,

$$\int_0^x (1+t^3)^{1/4}\,dt = \int_0^x\left(\sum_{n=0}^{\infty}\binom{1/4}{n}t^{3n}\right)dt$$

$$= \sum_{n=0}^{\infty}\binom{1/4}{n}\int_0^x t^{3n}\,dt = \sum_{n=0}^{\infty}\binom{1/4}{n}\frac{t^{3n+1}}{3n+1}\Big|_0^x = \sum_{n=0}^{\infty}\binom{1/4}{n}\frac{x^{3n+1}}{3n+1}.$$

4) First, we find the critical points of the function:

$$f_x = x^2 - 1$$

$$f_x = 0 \Rightarrow x^2 - 1 = 0 \Rightarrow x = \pm 1$$

$$f_y = y^2 - 1$$

$$f_y = 0 \Rightarrow y^2 - 1 = 0 \Rightarrow y = \pm 1$$

Therefore, there are four critical points: $(1,1)$, $(-1,-1)$, $(-1,1)$, $(1,-1)$.

Next, we find the second partial derivatives:

$$f_{xx} = 2x,\ f_{yy} = 2y \text{, and } f_{xy} = 0,$$

and form the function: $D(x,y) = f_{xx}(x,y) \cdot f_{yy}(x,y) - \left(f_{xy}(x,y)\right)^2$

$$D(x,y) = 4xy$$

Lastly, we classify the critical points by substituting each one into the $D(x,y)$ function and the $f_{xx}(x,y)$ function.

a) The point $(1,1)$:
$D(1,1) = 4 \cdot 1 \cdot 1 = 4 > 0$ and $f_{xx}(1,1) = 2 \cdot 1 = 2 > 0$
Thus $(1,1)$ is a relative minimum.

b) The point $(-1,-1)$:
$D(-1,-1) = 4 \cdot -1 \cdot -1 = 4 > 0$ and $f_{xx}(-1,-1) = 2 \cdot -1 = -2 < 0$
Thus $(1,1)$ is a relative maximum.

c) The point $(-1,1)$:
$D(-1,1) = 4 \cdot -1 \cdot 1 = -4 < 0$
Thus $(-1,1)$ is a saddle point.

d) The point $(1,-1)$:
$D(1,-1) = 4 \cdot 1 \cdot -1 = -4 < 0$
Thus $(1,-1)$ is a saddle point.

5)

a) Applying the root test, we get:

$$\lim_{n\to\infty}\sqrt[n]{|a_n|}=\lim_{n\to\infty}\frac{n^2+1}{2n^2+1}=\frac{1}{2}$$

Therefore the original series is absolutely convergent.

b) We apply the Integral Test. Consider the function $f(x)=\frac{\ln(x)}{x^{1.2}}$. This function is monotonic decreasing if $x>e^{\frac{5}{6}}$, and therefore we can use the Integral Test to analyze the convergence of this sequence. To compute $\int_1^\infty \frac{\ln(x)}{x^{1.2}}dx$, we use integration by parts.

Let $u=\ln(x)\Rightarrow du=\frac{1}{x}dx$ and $dv=x^{-1.2}dx\Rightarrow v=\int dv=\int x^{-1.2}dx=\frac{x^{-0.2}}{-0.2}$, so

$$\int\frac{\ln(x)}{x^{1.2}}dx=-\frac{x^{-0.2}}{0.2}\ln(x)+\frac{1}{0.2}\int x^{-1.2}dx=-\frac{x^{-0.2}}{0.2}\ln(x)-\frac{1}{(0.2)^2}x^{-0.2}$$

Therefore,

$$\int_1^\infty\frac{\ln(x)}{x^{1.2}}dx=\lim_{t\to\infty}\left[-\frac{x^{-0.2}}{0.2}\ln(x)-\frac{1}{(0.2)^2}x^{-0.2}\right]_1^t$$

$$=\lim_{t\to\infty}\left[-\frac{\ln(t)}{(0.2)t^{0.2}}-\frac{1}{(0.2)^2t^{0.2}}+\frac{\ln(1)}{0.2}+\frac{1}{(0.2)^2}\right]=25.$$

Since the integral converges, the series does as well.

c) This is an alternating series with $b_n=\frac{1}{n\ln(n)}$ that satisfies the two properties.

i) $b_{n+1}\le b_n$ since $\frac{1}{(n+1)\ln(n+1)}\le\frac{1}{n\ln(n)}$

ii) $\lim_{n\to\infty}b_n=\lim_{n\to\infty}\frac{1}{n\ln(n)}=0$

And therefore the series is convergent. However, the series is not absolutely convergent because the series of absolute values $a_n=\left|\frac{(-1)^n}{n\ln(n)}\right|=\frac{1}{n\ln(n)}$,

$f(x)=\frac{1}{x\ln(x)}$ diverges, as we can prove by using the integral test:

$$\int_2^\infty\frac{dx}{x\ln(x)}=\lim_{t\to\infty}\ln(\ln(x))\Big|_2^t=\infty$$

6) $\frac{dx}{dt} = x'(t) = 3 - 3t^2$ and $\frac{dy}{dt} = y'(t) = 1$.

$x'(t) = 0 \Leftrightarrow t = \pm 1$.

To find the intervals where the curve increasing or decreasing, we find the signs of $x'(t)$ and $y'(t)$:

t	$(-\infty, -1)$	$(-1, 1)$	$(1, \infty)$
$x'(t)$	–	+	–
$y'(t)$	+	+	+
x	←	→	←
y	↑	↑	↑

Thus $y(t)$ y(t) is strictly increasing, while $x(t)$ has a maximum at $t = 1$ and a minimum at $t = -1$.

For concavity, we calculate $\frac{d^2y}{dx^2}$:

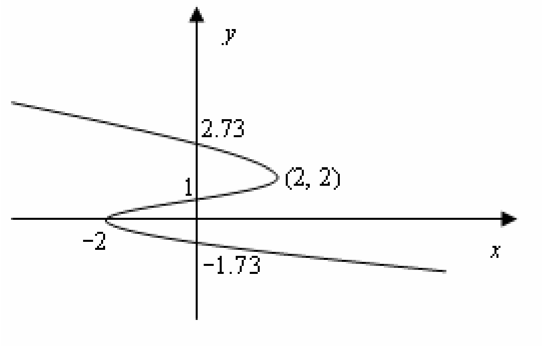
$$\frac{d^2y}{dx^2} = \frac{x'(t)y''(t) - y'(t)x''(t)}{[x'(t)]^3} = \frac{(3 - 3t^2)0 - 1(-6t)}{(3 - 3t^2)^3} = \frac{2t}{9(1 - t^2)^3}$$

t	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \infty)$
$\frac{d^2y}{dx^2}$	+	–	+	–
Curve C	Conc. Up	Conc. down	Conc. Up	Conc. down

Plot some special points: $t = \pm 1$ and $t = 0, \pm\sqrt{3}, -1$ (intercept points).

t	$-\sqrt{3}$	-1	0	1	$\sqrt{3}$
x	0	-2	0	2	0
y	$1 - \sqrt{3}$	0	1	2	$1+\sqrt{3}$

Combining these, we obtain the graph:



7) Let $y(t)$ be the amount of salt (in kg) present in the tank after t minutes

Let $t = 0$ be the time when a salt solution with a concentration of 0.1 kg/L starts running into the tank. The initial amount of salt in the tank is 30 kg, so $y(0) = 30$ (initial value condition).

The amount of salt per minute is run into the tank is (10 L/min)×0.1 kg/L = 1 kg/min.

The concentration of salt solution in a *well stirred* tank at any given time t is $\frac{y(t)}{800+4t}$ kg/L because the solution (brine) volume in the tank gains $(10 - 6) = 4$ litres per min. The amount of salt per minute flows out the tank is

$$(6\text{ L/min}) \times \frac{y}{800+4t}\text{ kg/L} = \frac{3y}{400+2t}\text{ kg/min.}$$

The rate of change in the amount of salt remaining in the tank ($\frac{dy}{dt}$) must be equal the rate of salt coming *into* the tank (10 kg/min) *minus* rate of salt flowing *out* of the tank ($\frac{y}{50}$ kg/min). In other words, $\frac{dy}{dt} = 1 - \frac{3y}{400+2t}$. This is first order linear differential equation with $p(t) = \frac{3}{400+2t}$ and $q(t) = 1$.

The solution is $y = \frac{1}{I(t)}\int I(t)q(t)dt$ where $I(t) = e^{\int p(t)dt}$.

$$\int p(t)dt = \int \frac{3}{400+2t}dt = \frac{3}{2}\ln|400+2t|.$$

$$I(t) = e^{\frac{3}{2}\ln|400+2t|} = (400+2t)^{3/2}$$

$$\int I(t)q(t)dt = \int (400+2t)^{3/2}dt = \frac{1}{2}\frac{(400+2t)^{5/2}}{5/2} = \frac{1}{5}(400+2t)^{5/2} + C$$

Hence,

$$y(t) = \frac{\frac{1}{5}(400+2t)^{5/2} + C}{(400+2t)^{3/2}} = \frac{1}{5}(400+2t) + C(400+2t)^{-3/2}.$$

With initial value condition

$$y(0) = 30 \Rightarrow 30 = \frac{1}{5}(400) + C(400)^{-3/2} \Rightarrow C = -4\times 10^5$$

The solution is $y(t) = \frac{2}{5}(200+t) - 4\times 10^5(400+2t)^{-3/2}$.

8)

a) Use the ratio test:

$$\lim_{n\to\infty}\frac{|x|^{n+1}/\ln(n+2)}{|x|^{n}/\ln(n+1)}=\lim_{n\to\infty}|x|\frac{\ln(n+1)}{\ln(n+2)}=|x|\lim_{n\to\infty}\frac{1/(n+1)}{1/(n+2)}$$

$$=|x|\lim_{n\to\infty}\frac{n+2}{n+1}=|x|\lim_{n\to\infty}\frac{1+2/n}{1+1/n}=|x|.$$

Hence, the series converges for $|x| < 1$ and diverges for $|x| > 1$.

For $x = 1$, $\sum_{n=1}^{\infty}\frac{1}{\ln(n+1)}$ is divergent by the comparison test, since $\frac{1}{\ln n}>\frac{1}{n}$, and $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.

For $x = -1$, $\sum_{n=1}^{\infty}\frac{(-1)^n}{\ln(n+1)}$ converges by alternating series test.

Therefore, the series converges for $-1 \le x < 1$.

b) $e^{x}=\sum_{n=0}^{\infty}\frac{x^n}{n!}$. Hence $e^{-x^2}=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{n!}$ and

$$\int_0^1 e^{-x^2}dx=\int_0^1\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{n!}=\sum_{n=0}^{\infty}\left(\left.\frac{(-1)^n x^{2n+1}}{(2n+1)n!}\right]_0^1\right)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)n!}.$$

Since this is an alternating series we must find n such that

$$\frac{1}{(2n+1)n!}\le 0.005\Rightarrow 200\le(2n+1)n!\Rightarrow n\ge 4.$$

Hence we have $\int_0^1 e^{-x^2}dx\cong 1-\frac{1}{3}+\frac{1}{10}-\frac{1}{42}=\frac{26}{35}\approx 0.74.$

c) $\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$ for $|x|<1$.

Hence, $\sum_{n=1}^{\infty}nx^{n-1}=\frac{d}{dx}\left(\frac{1}{1-x}\right)=\frac{1}{(1-x)^2}$ and , $\sum_{n=1}^{\infty}nx^{n}=\frac{x}{(1-x)^2}$.

So $\sum_{n=1}^{\infty}\frac{n}{3^n}=\frac{\frac{1}{3}}{\left(1-\frac{1}{3}\right)^2}=\frac{3}{4}$. Hence, $\sum_{n=3}^{\infty}\frac{n}{3^n}=\frac{3}{4}-\frac{1}{3}-\frac{2}{9}=\frac{7}{36}$.