MAT223 FINAL EXAM

AUGUST 2007

1.

a. False. If, for example, U and W are different 1-dimensional subspaces of $V = \mathbb{R}^2$, then $U \cup W$ is like an X, two intersecting lines. Since it is not the origin, a single line through the origin, or the entire plane, $U \cup W$ is not a subspace of V.

b. True. Since obviously $\operatorname{span}\{u+v, u+w, v+w\} \subseteq \operatorname{span}\{u, v, w\}$ and $\operatorname{span}\{u, v, w\}$ is 3-dimensional, we need only check that $\{u+v, u+w, v+w\}$ is linearly independent. But if $\lambda(u+v) + \mu(u+w) + \nu(v+w) = 0$, then $(\lambda+\mu)u + (\lambda+\nu)v + (\mu+\nu)w = 0$. By independence of $\{u, v, w\}$, $\lambda+\mu = \lambda+\nu = \mu+\nu = 0$. Finally, solving the system, we see that $\lambda = \mu = \nu = 0$.

c. True. Since every $u \in U$ is orthogonal to every $v \in U^\perp$, we must have $U \subseteq (U^\perp)^\perp$. And since $\dim (U^\perp)^\perp = n - \dim U^\perp = n - (n - \dim U) = \dim U$, we have $U = (U^\perp)^\perp$.

d. False. Gram-Schmidt can be used either with or with out normalization. If used with normalization, you do get the same orth*normal* set. If used without normalization then you don't get exactly the same set of vectors, but rather multiples of them. The phrase "Gram-Schmidt ortho*gonal*ization" suggests the process is used here without normalizing.

e. True. First of all, the set $\{X_1, \ldots, X_m, Y_1, \ldots, Y_k\}$ **is** orthogonal: the X_i are orthogonal to each other, as are the Y_j, and each X_i is orthogonal to each Y_j since $X_i \in U$ and $Y_j \in U^\perp$. And it is a basis because it is an orthogonal set of $m + k = \dim U + \dim U^\perp = n$ vectors.

f. False. It doesn't send the zero vector to itself, for example. (It isn't additive either.)

g. True. Recall that $\dim (U \cap W) + \dim (U + W) = \dim U + \dim W$. Since $U \neq W$, we have $U \subsetneqq U + W \subseteq V$, but $\dim U = 2$ and $\dim V = 3$, which forces $U + W = V$. Thus the dimension equation becomes $\dim (U \cap W) + 3 = 2 + 2$.

h. False. It would have rank n if there existed a solution **for every** B. But if you only assume it has a solution for a single B, as in this problem, you can say anything about the rank, it could any number from 0 to n.

i. True. A is diagonal so it wears its eigenvalues visibly on the diagonal: 2 and 4.

j. True. We have $A\,X = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 12 \end{bmatrix} = 4 \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, so X **is** an eigenvector of eigenvalue 4.

k. True. (You should just know this.)

l. True. The columns are 5 vectors in $\mathbb{R}^4$, since they are more than the dimension they must be dependent.

m. False. If $B = 0$, for example, $\operatorname{rank}(A\,B) = 0$ and rank A might very well be nonzero. (The opposite inequality is true, however: since $\operatorname{im} A\,B \subseteq \operatorname{im} A$, we have $\operatorname{rank}(A\,B) \leqslant \operatorname{rank} A$.)

n. True. We have $\dim(W^{\perp}) = n - \dim W$, $\dim(U^{\perp}) = n - \dim U$ and $\dim W \geqslant \dim U$.

o. False. The correct relation is $\det(u\,A) = u^n \det(A)$.

2.

a. $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Since the matrix is in reduced row-echelon form, its rank is just the number of non-zero rows, namely, 2.

b. $T(w, x, y, z) = (y, z)$. This T is obviously linear and onto (any vector in $\mathbb{R}^2$ can be written as (y, z) and $T(\text{anything}, \text{anything}, y, z) = (y, z)$.)

c. $\left(\operatorname{span}\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right\}\right)^{\perp}$. For this U, we have $U^{\perp} = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right\}$. U is non-zero since it contains, for instance, $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$.

d. We have $x^3 - x^2 - 2x = x(x^2 - x - 2) = x(x-2)(x+1)$, so the diagonal matrix $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ works.

e. Let see what kinds of matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ satisfy $A^2 = A$. We have $A^2 = \begin{bmatrix} a^2 + b\,c & (a+d)b \\ (a+d)c & d^2 + b\,c \end{bmatrix}$. One way for this to be A is for $a + d = 1$, $a^2 - a = d^2 - d = b\,c$. That can happen if $a = 1$, $d = 0$ or the other way around and one of b and c is 0. This is enough flexibility to find a basis:

$$\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}.$$

(This really is a basis because you can easily express the standard basis in terms of this one.)

3. The characteristic polynomial is

$$\begin{vmatrix} \lambda + 2 & 0 & -3 \\ 3 & \lambda - 1 & -3 \\ 4 & 0 & \lambda - 5 \end{vmatrix} = (\lambda - 1)\begin{vmatrix} \lambda + 2 & -3 \\ 4 & \lambda - 5 \end{vmatrix} = (\lambda - 1)(\lambda^2 - 3\,\lambda + 2),$$

which factors as $(\lambda - 1)^2\,(\lambda - 2)$. Let's find eigenvectors corresponding to $\lambda = 1$. They're solutions of the system:

$$\begin{aligned} -2\,x + 3\,z &= x \\ -3\,x + y + 3\,z &= y \\ -4\,x + 5\,z &= z \end{aligned}$$