PREP101 Your Competitive Edge

Math 133 Test 1 (U of T) Study Sheet

Mathematics of Finance

COMPOUND AMOUNT: $S = P(1+r)^n$

- S = compound amount, P = original principle, r = periodic interest rate, , n = number of compounding periods
- Compound interest: S – P
- The periodic interest rate is obtained by dividing the nominal rate by the number of interest periods per year

EFFECTIVE RATE:

$$r_e = \left(1+\frac{r}{n}\right)^n - 1$$

- r = nominal rate, n = number of times interest in compounded per year

EXAMPLE:
Money is invested for a length of two years. In the first year, the nominal rate is 15%, compounded monthly. In the second year, the nominal rate is 10%, compounded quarterly. To the nearest two decimal places, the effective rate is:

SOLUTION:
\$1 at effective rate j for 2 years will accumulate to $(1+j)^2$.

\$1 at the given rates for two years will accumulate to

$$\left(1+\frac{0.15}{12}\right)^{12}\left(1+\frac{0.10}{4}\right)^4$$

Then: $(1+j)^2 = \left(1+\frac{0.15}{12}\right)^{12}\left(1+\frac{0.10}{4}\right)^4$

$$\Rightarrow (1+j)^2 = 1.2812558$$

$$\Rightarrow j = (1.2812558)^{1/2} - 1 = 0.131925704 \approx 13.19\%$$

EFFECTIVE ANNUAL INTEREST RATE:

$$i = \left(1+\frac{r}{m}\right)^{\frac{m}{n}} - 1$$

- r = the nominal interest rate compounded m times per year.
- We're finding effective annual interest rate compounded n times per year

PRESENT VALUE:

$$P = S(1+r)^{-n}$$

- The present value of S is the principle P which must be invested at the periodic rate of r for n interest periods to accumulate S.

ANNUITIES (THEORY):

- A payment made at the end of its period is called an ordinary annuity. A payment made at the beginning of its period is called an annuity due. This can be thought of as an initial payment, followed by n-1 ordinary annuity payments for n-1 periods.

PRESENT VALUE OF AN ANNUITY

Ordinary Annuity:

$$A = R\cdot\frac{1-(1+r)^{-n}}{r} = R\cdot a_{n|r}$$

(where $a_{n|r} = \frac{1-(1+r)^{-n}}{r}$).

Annuity Due:

$$A = R\cdot\left(1+a_{n-1|r}\right)$$

R is the present amount of annuity per payment period, n the number of payment periods, r the interest rate per period and S the future amount of annuity per payment period.

FUTURE AMOUNT OF AN ANNUITY

Ordinary Annuity:

$$S = R\cdot\frac{(1+r)^n - 1}{r} = R\cdot s_{n|r}$$

(where $s_{n|r} = \frac{(1+r)^n - 1}{r}$).

Annuity Due:

$$S = R\cdot\left(S_{n-1|r} - 1\right)$$

AMORTIZATION OF LOANS (MORTGAGES)

Periodic Payment:

$$R = \frac{A}{a_{n|r}} = A\cdot\frac{r}{1-(1+r)^{-n}}$$

Principal outstanding at the beginning of the kth period:

$$Ra_{n-k+1|r} = R\cdot\frac{1-(1+r)^{-n+k-1}}{r}$$

Amount of Interest in the kth Payment:

$$Rra_{n-k+1|r}$$

Amount of Principal Contained in the kth Payment:

$$R\left[1 - ra_{n-k+1|r}\right]$$

Total Interest Paid:

$$R\left(n - a_{n|r}\right), \text{ or } nR - A$$

EXAMPLE:
Georgina buys a car. She pays \$1,500 as a down-payment, and she agrees to pay \$182.50 every month for 3 years. The interest rate on the loan is 15%, compounded monthly. What was the cash price of the car?

SOLUTION:
Let P denote the cash price of the car.
We know that the cash price of the car is the sum of the down-payment and the 36 monthly payments (that include the interest from the loan.)
The interest is given by 15 % divided by 12 months.

I.e. $i = \frac{15\%}{12} = 1.25\% = 0.0125$.

Thus:

$$P = 1500 + 182.50a_{36|.0125} = 1500 + 182.50\cdot\left(\frac{1-(1.0125)^{-36}}{0.0125}\right)$$

$$P = 1500 + 52646.63 = 6764.63$$

Therefore, the cash price of the car is \$6,764.63.

PRICE OF A BOND: $P = V\cdot(1+i)^{-n} + rVa_{n|i}$

P = the price of a bond with face value V, with n outstanding interest payments at the rate of r each; i is the current interest rate per semi-annual period.
We call r the coupon rate and i the yield.

EXAMPLE:
A \$5000 bond that pays interest at a nominal interest rate of 14% (compounded semiannually) is redeemable at par at the end of 10 years. Find the purchase price to yield 10% compounded semiannually.

SOLUTION:
The bond pays

$$Vr = 5000\cdot\left(\frac{14\%}{2}\right) = 5000\cdot(0.07) = 350$$

semiannually and \$5000 at the end of 10 years.

Note that n = 20 and $i = \frac{10\%}{2} = 0.05$.
Thus the purchase price would have to be:

$$P = 350a_{20|0.05} + 5000\cdot(1.05)^{-20} = 4361.77 + 1884.45 = 6246.22$$

Therefore, the purchase price would be \$6,246.22.

Matrix Algebra

Matrix Transpose: The transpose of an m × n matrix A, written A^T, is the n × m matrix obtained by interchanging the rows and the columns of A

SOLVING SYSTEMS OF LINEAR EQUATIONS WITH MATRICES

In a system of m linear equations with n variables, such that ai j is the coefficient of the jth variable in the ith equation, the m × n matrix A = (ai j) is called the **coefficient matrix**; the augmented matrix is the coefficient matrix written with the addition of the right side of the equations as the added (rightmost) column

Elementary row operations: Interchanging two rows, multiplying one row by a nonzero number, adding a multiple of one row to a different row.

A matrix in **Row Reduced Echelon form (RREF)** has: the first nonzero entry from the left of each nonzero row is a 1; each leading 1 is to the right of all leading 1's in the rows above it; each leading 1 is the only nonzero entry in its column; all zero rows are at the bottom.

Rank of a matrix: The number of leading 1's of a matrix in RREF

NUMBER OF SOLUTIONS OF A SYSTEM OF LINEAR EQUATIONS

- No solutions (if rank(Coefficient matrix) < rank(Augmented matrix)
- Number of parameters of a system of linear equations: # parameters = # variables − # of leading 1's of equivalent augmented matrix in RREF

EXAMPLE:
Determine the number of solutions of the following system of linear equations:

$$10x - 2y + z = 2$$
$$x + y + 3z = 17$$
$$15x + 3y + 16z = 87$$
$$9x - 3y + 2z = -15$$

SOLUTION:

Step 1. Create the augmented matrix:

$$\left[\begin{array}{ccc|c} 10 & -2 & 1 & 2 \\ 1 & 1 & 3 & 17 \\ 15 & 3 & 16 & 87 \\ 9 & -3 & 2 & -15 \end{array}\right]$$

Step 2. Reduce:

$$\left[\begin{array}{ccc|c} 10 & -2 & 1 & 2 \\ 1 & 1 & 3 & 17 \\ 15 & 3 & 16 & 87 \\ 9 & -3 & 2 & -15 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 3 & 17 \\ 10 & -2 & 1 & 2 \\ 15 & 3 & 16 & 87 \\ 9 & -3 & 2 & -15 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 1 & 3 & 17 \\ 0 & -12 & 29 & -168 \\ 0 & -12 & 29 & -168 \\ 0 & -12 & 29 & -168 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 3 & 17 \\ 0 & 1 & -29/12 & 14 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We have 3 variables and 2 leading 1's. Therefore, we must have a one-parameter family of solutions. There are infinitely many solutions.

Gaussian algorithm: To put a matrix in RREF

Step 1. If the matrix consists entirely of zeros, stop – it is already in row-echelon form.

Step 2. Find the first column from the left containing a nonzero entry (call it a), and move the row containing that entry to the top position.

Step 3. Multiply that row by 1/a to create a leading 1.

Step 4. By subtracting multiples of that row from the rows below it (and those above it if possible), make each entry below (and above) the leading 1 zero.

Step 5. Repeat steps 1-4 on the matrix consisting of the remaining rows. The process stops when either no rows remain or the remaining rows consist entirely of zeros.

EXAMPLE:

Solve the following system of equations by using the technique of Gauss-Jordan elimination:

x + 3y –2 = 0
x – 4z –4 = 0
2x + 2y +8z –8 = 0

SOLUTION:

Step 1. Write the system into standard form and create the augmented matrix

$$\begin{aligned} x+3y+0z&=2\\ x+0y-4z&=4\\ 2x+2y+8z&=8 \end{aligned} \Rightarrow \left[\begin{array}{ccc|c} 1 & 3 & 0 & 2\\ 1 & 0 & -4 & 4\\ 2 & 2 & 8 & 8 \end{array}\right]$$

Step 2. Solve using EROS.

1.Make the entry a_{11} a leading 1 for column 1.

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & 2\\ 1 & 0 & -4 & 4\\ 2 & 2 & 8 & 8 \end{array}\right] \xrightarrow{R1\leftrightarrow R2} \left[\begin{array}{ccc|c} 1 & 0 & -4 & 4\\ 1 & 3 & 0 & 2\\ 2 & 2 & 8 & 8 \end{array}\right]$$

$$\xrightarrow{R2-R1} \left[\begin{array}{ccc|c} 1 & 0 & -4 & 4\\ 0 & 3 & 4 & -2\\ 2 & 2 & 8 & 8 \end{array}\right]$$

$$\xrightarrow{R3-2R1} \left[\begin{array}{ccc|c} 1 & 0 & -4 & 4\\ 0 & 3 & 4 & -2\\ 0 & 2 & 16 & 0 \end{array}\right] \xrightarrow{R2\leftrightarrow R3*\frac{1}{2}} \left[\begin{array}{ccc|c} 1 & 0 & -4 & 4\\ 0 & 1 & 8 & 0\\ 0 & 3 & 4 & 2 \end{array}\right]$$

Make entry a_{22} into a leading 1 for column 2.

$$\left[\begin{array}{ccc|c} 1 & 0 & -4 & 4\\ 0 & 1 & 8 & 0\\ 0 & 3 & 4 & 2 \end{array}\right] \xrightarrow{R3-3*R2} \left[\begin{array}{ccc|c} 1 & 0 & -4 & 4\\ 0 & 1 & 8 & 0\\ 0 & 0 & -20 & 2 \end{array}\right]$$

$$\xrightarrow{R3*\frac{-1}{20}} \left[\begin{array}{ccc|c} 1 & 0 & -4 & 4\\ 0 & 1 & 8 & 0\\ 0 & 0 & 1 & 1/10 \end{array}\right]$$

2.Make the entry a_{33} into a leading 1 for column 3

$$\left[\begin{array}{ccc|c} 1 & 0 & -4 & 4\\ 0 & 1 & 8 & 0\\ 0 & 0 & 1 & 1/10 \end{array}\right] \xrightarrow{R2-8*R3} \left[\begin{array}{ccc|c} 1 & 0 & -4 & 4\\ 0 & 1 & 0 & -8/10\\ 0 & 0 & 1 & 1/10 \end{array}\right]$$

$$\xrightarrow{R1+4*R3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 44/10\\ 0 & 1 & 0 & -8/10\\ 0 & 0 & 1 & 1/10 \end{array}\right]$$

Our matrix is now in reduced row echelon form.

Step 3. Read off the solution set.

$x = 44/10$, $y = -8/10$, $z = 1/10$

Matrix Inverse: For square matrices A and B: if AB = BA = I, then $B = A^{-1}$.

Inversion algorithm: To find the inverse of a given matrix

Step 1. Form the augmented matrix [A|I] (for square matrix A and the Identity matrix of the same order/dimension).

Step 2. Use elementary row operations on matrix [A|I] to transform the A section of the [A|I] matrix to row-reduced echelon form.

Step 3. Let [C|D] be the final augmented matrix in RREF.

Step 4. Then if C = I then $A^{-1} = D$. If C is not the identity matrix, then A is not invertible (i.e. it is singular).

EXAMPLE:

Find the inverse of Matrix M where $M = \begin{bmatrix} 1 & 1 & -1\\ 2 & 2 & 1\\ 1 & 2 & 1 \end{bmatrix}$.

SOLUTION:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0\\ 2 & 2 & 1 & 0 & 1 & 0\\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0\\ 0 & 0 & 3 & -2 & 1 & 0\\ 0 & 1 & 2 & -1 & 0 & 1 \end{array}\right] \rightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0\\ 0 & 1 & 2 & -1 & 0 & 1\\ 0 & 0 & 3 & -2 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 2 & 0 & -1\\ 0 & 1 & 2 & -1 & 0 & 1\\ 0 & 0 & 3 & -2 & 1 & 0 \end{array}\right] \rightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 2 & 0 & -1\\ 0 & 1 & 2 & -1 & 0 & 1\\ 0 & 0 & 1 & -2/3 & 1/3 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -1\\ 0 & 1 & 0 & 1/3 & -2/3 & 1\\ 0 & 0 & 1 & -2/3 & 1/3 & 0 \end{array}\right]$$

Therefore, the inverse of Matrix M is

$$M^{-1} = \begin{bmatrix} 0 & 1 & -1\\ 1/3 & -2/3 & 1\\ -2/3 & 1/3 & 0 \end{bmatrix}$$

METHOD OF INVERSES:

Suppose a system of n equations in n variables is written in matrix form as AX = B. If the $n \times n$ coefficient matrix A is invertible, the system has the unique solution $X = A^{-1}B$.

DETERMINANTS:

detA = $\sum_{j=1}^{n} a_{1j}C_{1j}$, for Cij, the (i, j)-cofactor of A.

The determinant A is denoted by $|A|$

EXAMPLE:

Find the determinant of matrix

$$A = \begin{bmatrix} 3 & 2 & 4\\ 2 & 4 & 1\\ 2 & 2 & 2 \end{bmatrix}$$

SOLUTION:

detA = $\sum_{j=1}^{n} a_{1j}C_{1j}$.

$$\det(A) = (-1)^{1+1}\cdot(3)\cdot\det\begin{bmatrix} 4 & 1\\ 2 & 2 \end{bmatrix}$$

$$+(-1)^{1+2}\cdot(2)\cdot\det\begin{bmatrix} 2 & 1\\ 2 & 2 \end{bmatrix} + (-1)^{1+3}\cdot(4)\cdot\det\begin{bmatrix} 2 & 4\\ 2 & 2 \end{bmatrix}$$

$$\det(A) = 3\cdot(8-2) - 2(4-2) + 4(4-8)$$

$$\det(A) = 3\cdot 6 - 2\cdot 2 + 4\cdot(-4)$$

$$\det(A) = -2$$

CRAMER'S RULE:

For $A\vec{x} = \vec{b}$, a system of linear equations in variables $x_1, x_2, ..., x_n$ such that A is an invertible $n \times n$ matrix and $|A| \neq 0$. The (unique) solution to the system is given by:

$$x_1 = \frac{\det A(1)}{|A|}, x_2 = \frac{\det A(2)}{|A|}, \ldots, \frac{\det A(n)}{|A|}$$

; we construct A(i) by deleting the ith column of the coefficient matrix, and replacing it with the column vector $\vec{b}$.

EXAMPLE:

Use Cramer's Rule to find the solution to the following system of equations.

2x + y - z = 1
x + y + z = 7
x + 2y - z = 0

SOLUTION:

Step 1. Rewrite the SLE in matrix notation.

$$A = \begin{bmatrix} 2 & 1 & -1\\ 1 & 1 & 1\\ 1 & 2 & -1 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x\\ y\\ z \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 1\\ 7\\ 0 \end{bmatrix}$$

Step 2. Ensure that the system has a unique solution.

$\det(A) = -5 \neq 0$

Step 3. Construct A(1), A(2) and A(3)

$$A(1) = \begin{bmatrix} 1 & 1 & -1\\ 7 & 1 & 1\\ 0 & 2 & -1 \end{bmatrix}, \quad A(2) = \begin{bmatrix} 2 & 1 & -1\\ 1 & 7 & 1\\ 1 & 0 & -1 \end{bmatrix},$$

$$A(3) = \begin{bmatrix} 2 & 1 & 1\\ 1 & 1 & 7\\ 1 & 2 & 0 \end{bmatrix}$$

Step 4. Compute the determinant of matrices A(1), A(2) and A(3)

$\det(A(1)) = -10$, $\det(A(2)) = -5$, $\det(A(3)) = -20$

Step 5. Solve for the variables x, y, z:

$$x = \frac{\det(A(1))}{\det(A)} = \frac{-10}{-5} = 2,$$

$$y = \frac{\det(A(2))}{\det(A)} = \frac{-5}{-5} = 1,$$

$$z = \frac{\det(A(3))}{\det(A)} = \frac{-20}{-5} = 4$$