PREP101
Your Competitive Edge

Math 135 (U of T) Study Sheet

Review

LINEAR QUADRATIC EQUATION:

for $ax^2+bx+c=0$, $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

LOGARITHMS:

$\log_b(x\cdot y)=\log_b(x)+\log_b(y)$,

$\log_b\left(\dfrac{x}{y}\right)=\log_b(x)-\log_b(y)$, $\log_b(x^n)=n\cdot\log_b(x)$

EXPONENT LAWS:

$b^m\cdot b^n=b^{m+n}$, $\left(\dfrac{b^m}{b^n}\right)=b^{m-n}$, $(b^m)^n=b^{m\cdot n}$, $b^1=b$, $b^0=1$

TRIGONOMETRY:

$\sin^2u+\cos^2u=1$, $1+\tan^2u=\sec^2u$,

$1+\cot^2u=\csc^2u$,

$\sin(u\pm v)=\sin u\cos v\pm\cos u\sin v$,

$\cos(u\pm v)=\cos u\cos v\mp\sin u\sin v$,

$\tan(u\pm v)=\dfrac{\tan u\pm\tan v}{1\mp\tan u\tan v}$, $\sin(2u)=2\sin u\cos u$,

$\cos(2u)=\cos^2u-\sin^2u=2\cos^2u-1=1-2\sin^2u$

, $\tan(2u)=\dfrac{2\tan u}{1-\tan^2u}$, $\sin^{-1}x+\cos^{-1}x=\dfrac{\pi}{2}$,

$\tan^{-1}x+\cot^{-1}x=\dfrac{\pi}{2}$, $\sec^{-1}x+\csc^{-1}x=\dfrac{\pi}{2}$,

$\csc^{-1}(x)=\sin^{-1}\left(\dfrac{1}{x}\right)$, $\sec^{-1}x=\cos^{-1}\left(\dfrac{1}{x}\right)$, $\cot x=\tan^{-1}\left(\dfrac{1}{x}\right)$

Limits and Continuity

The function *f(x)* has limit *L,* as *x* approaches *a*, denoted $\lim_{x\to a}f(x)=L$ if given any $\varepsilon>0$, there exists a $\delta>0$ such that |*f*(*x*) - *L*| < ε for all *x* satisfying 0 < |*x* – *a*|< δ.

$$\lim_{x\to a}[f(x)\pm g(x)]=\lim_{x\to a}f(x)\pm\lim_{x\to a}g(x),$$

$$\lim_{x\to a}[f(x)\cdot g(x)]=\lim_{x\to a}f(x)\cdot\lim_{x\to a}g(x),$$

$$\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right]=\frac{\lim_{x\to a}f(x)}{\lim_{x\to a}g(x)},\ \left(\lim_{x\to a}g(x)\neq 0\right),$$

$$\lim_{x\to a}[f(x)]^n=\left[\lim_{x\to a}f(x)\right]^n$$

TECHNIQUES FOR FINDING LIMITS:

- First try substitution, factoring
- If you’re taking the limit as $x\to\infty$ of a quotient, and substitution yields an invalid answer, then try first to divide both the numerator and the denominator of the limit expression by its highest power of *x*
- If the limit expression contains ***absolute values***, then try breaking the limit up into *two one-sided limits*.
- If the limit expression is a quotient, and if factoring doesn’t work, then try l’Hopital’s Rule

L’HOPITAL’S RULE:

If the limit of the quotient of differentiable functions $f(x)$ and $g(x)$ are of types $\frac{0}{0}$ or $\frac{\infty}{\infty}$, and if $g'(a)$ is not 0, then $\lim_{x\to a}\dfrac{f(x)}{g(x)}=\lim_{x\to a}\dfrac{f'(x)}{g'(x)}$

EXAMPLE:

Determine the following limit: $\lim_{x\to\infty}\dfrac{3+x^2-6x^3}{5x^3-4x+2}$.

SOLUTION:

Divide the numerator and the denominator by the highest power of *x* (in this case, x^3).

$$\lim_{x\to\infty}\frac{3+x^2-6x^3}{5x^3-4x+2}$$

$$=\lim_{x\to\infty}\frac{x^3\cdot\left(\frac{3}{x^3}+\frac{x^2}{x^3}-\frac{6x^3}{x^3}\right)}{x^3\cdot\left(\frac{5x^3}{x^3}-\frac{4x}{x^3}+\frac{2}{x^3}\right)}=\lim_{x\to\infty}\frac{\frac{3}{x^3}+\frac{1}{x}-6}{5-\frac{4}{x^2}+\frac{2}{x^3}}$$

The terms $\frac{3}{x^3}$ and $\frac{1}{x}$ in the numerator and the terms $\frac{4}{x^2}$ and $\frac{2}{x^3}$ in the denominator tend toward zero as *x* approaches infinity.

Therefore, we are left with:

$$\lim_{x\to\infty}\frac{3+x^2-6x^3}{5x^3-4x+2}=\lim_{x\to\infty}\frac{0+0-6}{5-0+0}=\frac{-6}{5}.$$

The Derivative

The derivative of $f(x)$ at $x=a$, $f'(a)$, represents the slope of the tangent line to $f(x)$ at $x=a$.

RULES FOR DIFFERENTIATION: for functions *u*,*v* and for constants *c*, *n*: $\dfrac{d}{dx}(cu^n)=cnu^{n-1}\dfrac{du}{dx}$,

$\dfrac{d}{dx}(uv)=u\dfrac{dv}{dx}+v\dfrac{du}{dx}$, $\dfrac{d}{dx}\left(\dfrac{u}{v}\right)=\dfrac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$

DERIVATIVES TO REMEMBER:

$\dfrac{d}{du}(\ln u)=\dfrac{1}{u}\cdot\dfrac{du}{dx}$, $\dfrac{d}{dx}(e^u)=e^u\dfrac{du}{dx}$,

$\dfrac{d}{dx}(a^u)=a^u(\ln a)\dfrac{du}{dx}$, $\dfrac{d}{dx}(\sin x)=\cos x$,

$\dfrac{d}{dx}(\cos x)=-\sin x$, $\dfrac{d}{dx}(\tan x)=\sec^2x$,

$\dfrac{d}{dx}(\cot x)=-\csc^2x$, $\dfrac{d}{dx}(\sec x)=\sec x\cdot\tan x$,

$\dfrac{d}{dx}(\csc x)=-\csc x\cdot\cot x$, inverse trig ones

CHAIN RULE: If $f(x)=(a\circ b)(x)$, then $f'(x)=a'(b(x))\cdot b'(x)$

DIFFERENTIATION BY FIRST PRINCIPLES:

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$

EXAMPLE:

Find the derivative of the function $f(x)=(x+1)^2$ by first principles.

SOLUTION:

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{(x+h+1)^2-(x+1)^2}{h}$$

$$=\lim_{h\to 0}\frac{(x^2+2xh+2x+h^2+2h+1)-(x^2+2x+1)}{h}$$

$$=\lim_{h\to 0}\frac{2xh+h^2+2h}{h}=\lim_{h\to 0}\frac{h(2x+h+2)}{h}$$

$$=\lim_{h\to 0}2x+h+2=2(x+1)$$

EXAMPLE:

Differentiate the function $y=\dfrac{2}{(2x^3+x)^2}$.

SOLUTION:

First rewrite the function as follows:

$$y=\frac{2}{(2x^3+x)^2}=2(2x^3+x)^{-2}$$

Let $u=2x^3+x\Rightarrow u'=6x^2+1$.

Our function becomes $y=2u^{-2}$.

Then $y'=-4u^{-3}\cdot u'$.

$\Rightarrow y'=-4\cdot(2x^3+x)^{-3}\cdot(6x^2+1)$

Power rule: $\dfrac{d}{dx}c^{u(x)}=c^{u(x)}\cdot\ln c\cdot\dfrac{du}{dx}$

EXAMPLE:

Differentiate the following equation with respect to *x*:

$f(x)=\sqrt{4^{x-3}}$.

SOLUTION:

First, we will rewrite the given equation as follows:

$$f(x)=\sqrt{4^{x-3}}=4^{\frac{x-3}{2}}=4^{\frac{x}{2}-\frac{3}{2}}$$

Now we use the Power Rule to solve:

$$f'(x)=\left(4^{\frac{x}{2}-\frac{3}{2}}\right)\cdot\ln 4\cdot\frac{d}{dx}\left(\frac{x}{2}-\frac{3}{2}\right)=\left(4^{\frac{x}{2}-\frac{3}{2}}\right)\cdot\ln 4\cdot\left(\frac{1}{2}\right)$$

Higher order derivatives: $f''(x)=\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$.

EXAMPLE:

Find *f''*(2) if *f*(*x*) = *x* ln (1 + x^2).

SOLUTION:

First take the first derivative:

$$f'(x)=\ln(1+x^2)+x\cdot\left(\frac{1}{1+x^2}\right)\cdot\frac{d}{dx}(1+x^2)$$

$$=\ln(1+x^2)+x\cdot\left(\frac{1}{1+x^2}\right)\cdot(2x)=\ln(1+x^2)+\frac{2x^2}{1+x^2}$$

Take the derivative of $f'(x)$ as follows:

$$f''(x)=\frac{d}{dx}\left(\ln(1+x^2)+\frac{2x^2}{1+x^2}\right)$$

$$=\frac{1}{1+x^2}\frac{d}{dx}(1+x^2)+\frac{4x\cdot(1+x^2)-4x^3}{(1+x^2)^2}$$

$$=\frac{2x}{1+x^2}+\frac{4x\cdot(1+x^2)-4x^3}{(1+x^2)^2}=\frac{6x+2x^3}{(1+x^2)^2}$$

Techniques of differentiation

IMPLICIT DIFFERENTIATION:

to differentiate an implicit function

Step 1. Take the derivative of both sides with respect to *x*. Use the Chain Rule on terms involving *y* (and note that the derivative of y with respect to *x* must be left as dy/dx.)

Step 2. Collect all terms involving dy/dx on one side of the equation.

Step 3. Solve for dy/dx.

LOGARITHMIC DIFFERENTIATION:
we must use this form of differentiation when our function that we're differentiating has an x both in the base and in the exponent. We could also use this form of differentiation to differentiate a function with complicated exponent/a product of several functions, etc.
Step 1. Take the 'ln' of both sides (to find an expression of the form $\ln y = \ln[f(x)]$).
Step 2. Simplify $\ln(f(x))$ by using the properties of logarithms.
Step 3. Differentiate both sides with respect to *x* (thus: $\frac{1}{y}\cdot\frac{dy}{dx} = \frac{d}{dx}(\ln f(x))$).
Step 4. Solve for dy/dx.
Step 5. Express the answer in terms of *x* only (substitution *f(x)* for *y*).
EXAMPLE:
Find $\frac{dy}{dx}$ for the following expression:

$$y = (1+x)^{\cos x}.$$

SOLUTION:
Since there is an x both in the base and in the exponent, we use logarithmic differentiation. Take the ln of both sides of the equation, and then simplify using the properties of the ln function:

$$y = (1+x)^{\cos x} \Rightarrow \ln y = \ln(1+x)^{\cos x}$$
$$\Rightarrow \ln y = \cos(x)\cdot\ln(1+x)$$

Differentiate both sides:

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(\cos(x)\cdot\ln(1+x))$$
$$\Rightarrow \frac{y'}{y} = -\sin(x)\ln(1+x) + \cos(x)\cdot\frac{1}{1+x}$$
$$\Rightarrow y' = y\left[-\sin(x)\ln(1+x) + \frac{1}{1+x}\cos(x)\right]$$

Substitute the function $y = (1+x)^{\cos x}$ back into the derivative equation:

$$\Rightarrow y' = (1+x)^{\cos(x)}\cdot\left[-\sin(x)\ln(1+x) + \frac{1}{1+x}\cos(x)\right]$$

Applications of the Derivative

CRITICAL POINTS: The values of $X \in$ domain of *f(x)* such that $f'(x) = 0$ or $f'(x)$ is not defined.

FIRST DERIVATIVE TEST: If $f'(p) = 0$ and *f '* changes from negative to positive at *p*, then *f* has a relative minimum at *p*. If *f '* changes from positive to negative at *p*, then *f* has a relative maximum at *p*.

SECOND DERIVATIVE TEST: If $f'(p) = 0$ and $f''(p) > 0$, then *f* has a relative minimum at *p*. If $f'(p) = 0$ and $f''(p) < 0$, then *f* has a relative maximum at *p*. If $f'(p) = 0$ and $f''(p) = 0$, then the test fails.

CONCAVITY: If $f''(p) > 0$ on an interval, then *f*(*x*) is concave up on that interval; if $f''(p) < 0$, then *f*(*x*) is concave down on that interval. A point of inflection occurs when $f''(p)$ changes sign (and thus concavity).

VERTICAL ASYMPTOTE: The line *x* = *a* is a ***vertical asymptote*** for the graph of the function *f(x)* if and only if $\lim_{x\to a+} f(x) = \pm\infty$ or $\lim_{x\to a-} f(x) = \pm\infty$.

HORIZONTAL ASYMPTOTE: The line *x* = *b* is a ***horizontal asymptote*** for the graph of the function *f(x)* if and only if $\lim_{x\to+\infty} f(x) = b$ or $\lim_{x\to-\infty} f(x) = b$

CURVE SKETCHING
Step 1. Find the intercepts.
Step 2. Find all the asymptotes.
Step 3. Find critical points and the intervals of increase/decrease.
Step 4. Find inflection points and the intervals in which the function is concave up/down.
EXAMPLE:
Sketch the curve $f(x) = \frac{14}{2x^2+7}$, showing all of the usual properties.
SOLUTION:
Note that the function *f* is even and positive for all *x*.
(i) Intercepts.
The graph of *f* has a *y*-intercept at *y* = 2 and has no *x*-intercept.
(ii) Asymptotes.
As $x \to \pm\infty$, $f(x) \to 0$: Hence *y* = 0 is the H.A.
There is no V.A.
(iii) Maxima/minima and intervals of increase and decrease.

$$f'(x) = 14(-1)\frac{(4x)}{(2x^2+7)^2} = \frac{-56x}{(2x^2+7)^2}.$$
$$f'(x) = 0 \Leftrightarrow x = 0.$$

x	f'	f
$(-\infty,0)$	+	Increasing
$(0,\infty)$	–	Decreasing

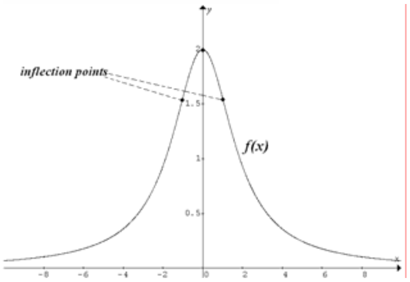
(iv) Concavity and points of inflection.

$$f''(x) = -56\frac{(2x^2+7)^2 - 2x(2x^2+7)4x}{(2x^2+7)^4} = 56\frac{(6x^2-7)}{(2x^2+7)^3}$$

So,

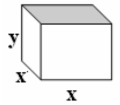
$$f''(x) = 0 \Leftrightarrow x = \pm\sqrt{\frac{7}{6}},\ \text{these are the inflection points.}$$

x	f'	f
$\left(-\infty,-\sqrt{\frac{7}{6}}\right)$	+	Concave up
$\left(-\sqrt{\frac{7}{6}},\sqrt{\frac{7}{6}}\right)$	–	Concave down
$\left(\sqrt{\frac{7}{6}},\infty\right)$	+	Concave up



OPTIMIZATION PROBLEMS
Step 1. Determine what we're trying to maximize/minimize and write an equation for this.
Step 2. Write a second equation from additional information given in the problem, isolate one of the two variables, and substitute this into the first equation.
Step 3. Take the derivative of this equation, set it to zero and solve for the remaining variable.
Step 4. Plug the value of this variable into the original equations to solve for the remaining variables.

EXAMPLE:
A box with a square bottom, vertical sides and no top is to be built so as to contain 16 cubic metres of liquid. If the cost of the bottom material is \$4 per square metre and the cost of the side material is \$1 per square metre, find the dimensions which will give the lowest cost.
SOLUTION:



The volume of the cube is given by $V = x^2y = 16\text{ m}^3$.
The cost of bottom is $x^2\times(\$4) = \$4x^2$.
The cost of side material is $4\times(xy)\times(\$1) = \$4xy$
The total cost is $C = 4x^2 + 4xy$.
$C(x) = 4x^2 + 4x(16/x^2) = 4x^2 + \frac{64}{x}$.

$$C'(x) = 8x - 64x^{-2} = 8\left(x - \frac{8}{x^2}\right).$$
$$= 8\left(\frac{x^3-8}{x^2}\right) = 8\frac{(x-2)(x^2+2x+4)}{x^2}$$

X	C'	C
$(0, 2)$	-	Dec.
$(2, \infty)$	+	Inc.

As $x \to \infty$, $C(x) \to \infty$.
As $x \to 0$, $C(x) \to \infty$.
Hence, the local minimum of *C* at *x* = 2 is also the absolute min.
Hence, *x* = 2 m and $y = 16/x^2 = 4$ m.

TANGENT PROBLEMS/WORD PROBLEMS
EXAMPLE:
Find the equation of the tangent line of maximum slope to the curve $y = 1 + 2x - x^3$.
SOLUTION:
The slope of tangent line *m*(*x*) = $y' = \frac{dy}{dx} = 2 - 3x^2$.
To maximize m, set $m'(x) = 0$.

$$m'(x) = -6x.$$
$$-6x = 0 \Rightarrow x = 0$$

x	M'	m
$(-\infty, 0)$	+	Increasing
$(0, \infty)$	-	Decreasing

From the table, *m*(*x*) is maximum when *x* = 0 and *m*(0) = 2.
At *x* = 0, *y* = 1.
Therefore, the equation of the tangent line of maximum slope is $y - 1 = 2\cdot(x-0) \Rightarrow y = 2x + 1$.