MATH 137 (U of T) Study Sheet

Properties of Real Numbers

- Properties of real numbers: associativity, commutativity, existence of identity, existence of inverses
- Subsets of real numbers: natural numbers (e.g. 1,2,3,…), integers (e.g. -2,-1,0,1,2,…), rational numbers (i.e. $x \,\Big|\, x = \dfrac{p}{q}$ for *p*, *q* integers, $q \neq 0$).

Properties of Functions

Domain is the set of values of ***x*** for which ***f(x)*** is defined

Range is the set of all possible values ***f(x)***

Even functions are functions that have the property $f(-x) = f(x)$ for all *x*

Odd functions are functions that have the property $f(-x) = -f(x)$ for all *x*

Invertible function: a function *f* is invertible if there exists a function *g* such that $y = f(x)$ if and only if $g(y) = x$

Increasing function: a function is increasing on an interval *I* if for all *x*, $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ on *I*

Decreasing function: a function is decreasing on an interval *I* if for all *x*, $f(x_1) > f(x_2)$ whenever $x_1 < x_2$ on *I*

EXAMPLE:

Suppose *f*(*x*) is even and

$$g(x) = \frac{x^2}{f(x) - 5f(x^3)}$$

. Determine whether *g*(*x*) is even, odd, or neither.

SOLUTION:

$$g(-x) = \frac{(-x)^2}{f(-x) - 5f([-x]^3)}$$

$$= \frac{x^2}{f(-x) - 5f(-x^3)} = \frac{x^2}{f(x) - 5f(x^3)}$$

since $f(x)$ is even.

Therefore, since $g(x) = g(-x)$, $g(x)$ is even.

Mathematical Induction

INDUCTION STEP BY STEP

Step 1—***Base case***. Show that the statement holds in the simplest case (normally for $n = 0$ and/or $n = 1$).

Step 2—***Induction Hypothesis***. Assume the statement holds for an arbitrary *k*.

Step 3. Prove that it holds for $k + 1$ given the above induction hypothesis.

Strong (or complete) induction: assume that the statement holds for $n < k$, and then prove it true for $n = k$.

EXAMPLE:

Prove that $1 + 2 + \ldots + n = \frac{n(n+1)}{2}$ (i.e. the sum of the first n positive integers is equal to $\frac{n(n+1)}{2}$.)

SOLUTION:

We'll use induction to prove the above equality.

Base case:

When n = 1, the left side of the equality is equal to 1.

When n = 1, the right side of the equality is equal to $\frac{1 \cdot (1+1)}{2} = \frac{2}{2} = 1$.

Thus, the left side is equal to the right side and the expression is valid for the case n = 1.

Induction Step:

Our induction hypothesis is that the expression holds for the case n = k, and thus the following expression holds:

$$1 + 2 + \ldots + k = \frac{k \cdot (k+1)}{2}$$

We now need to show the statement is true for *n* = *k* + 1.

That means, we are concerned with the sum of the first k + 1 integers, i.e. with the sum $1 + 2 + \ldots + k + k + 1$.

But we can write this as follows:

$$1 + 2 + \ldots + k + k + 1 = (1 + 2 + \ldots + k) + k + 1$$

But since our induction hypothesis is $1 + 2 + \ldots + k = \frac{k \cdot (k+1)}{2}$, we can re-write it as follows:

$$(1 + 2 + \ldots + k) + k + 1 = \frac{k(k+1)}{2} + (k+1)$$

Collecting the (k + 1) terms, we find:

$$(1 + 2 + \ldots + k) + k + 1 = \frac{k(k+1)}{2} + (k+1)$$

$$= (k+1) \cdot \left(\frac{k}{2} + 1\right) = \frac{(k+1) \cdot (k+2)}{2}$$

$$= (k+1)\frac{((k+1)+1)}{2}$$

Therefore, we know:

$$(1 + 2 + \ldots + k) + k + 1 = (k+1)\frac{((k+1)+1)}{2}$$

This is what we need to show. Therefore, the statement is true for all positive integers *n*.

Inequalities and Absolute Values

Absolute Value Rules: $|a + b| \leq |a| + |b|$, $|a - b| \geq \big||a| - |b|\big|$, $|a \cdot b| = |a| \cdot |b|$

Upper/lower bound: *A* has an ***upper bound*** if there is a number *a* such that for all $x \in A$, $x \leq a$. *A* has a lower bound if there is a number *a* such that for all $x \in A$, $a \leq x$.

Bounded: *A* is ***bounded*** *iff A* has either an upper bound and a lower bound; *iff* there exists a number *b* such that for all $x \in A$, $|x| \leq b$.

Least Upper Bound: *b* is the ***least upper bound*** of *A iff b* is an upper bound of *A* ***and*** if for all other upper bounds *a* of *A*, $b \leq a$.

Greatest Lower Bound: *b* is the ***greatest lower bound*** of *A iff b* is a lower bound of *A* ***and*** if for all lower bounds *a* of *A*, $a \leq b$.

Completeness of real numbers:

- If *A* is a non empty set of real numbers and *A* has an upper bound then *A* has a least upper bound
- If *A* is a non empty set of real numbers and *A* has a lower bound then *A* has a greatest lower bound

EXAMPLE:

Solve the inequality $x^2 + 7x - 8 < 0$.

SOLUTION:

First, lets factor the expression $x^2 + 7x - 8$ as follows: $x^2 + 7x - 8 = (x+8)(x-1)$.

This gives us: $(x+8)(x-1) < 0$.

For $x < -8$, both $x + 8$ and $x - 1$ are negative and, therefore $F(x) = (x+8)(x-1)$ is positive.

When we pass through *x* = –8, *x* + 8 changes sign and, therefore, so does *F*(x).

But when we later pass through *x* = 1, *x* – 1 changes sign and *F*(x) changes back to being positive.

Thus, *F*(x) is negative for –8 < x < 1.

Logarithmic Functions

Properties of logarithms:

$$\log_b(mn) = \log_b m + \log_b n,$$

$$\log_b\left(\frac{m}{n}\right) = \log_b m - \log_b n,$$

$$\log_b m^r = r\log_b m, \quad \log_b \frac{1}{m} = -\log_b m,$$

$$\log_b 1 = 0, \quad \log_b b = 1, \quad \log_b b^r = r,$$

$$b^{\log_b m} = m, \quad 10^{\log x} = x, \quad e^{\ln x} = x,$$

$$\log_b m = \frac{\log_a m}{\log_a b}.$$

EXAMPLE:

Solve for *x*: $\ln(x^2 + x) = 1 + \ln x$

SOLUTION:

$\ln(x^2 + x) = 1 + \ln x$

$\ln(x^2 + x) - \ln x = 1$

$$\ln\left(\frac{x^2 + x}{x}\right) = 1$$

$\ln(x + 1) = 1 \Rightarrow x + 1 = e \Rightarrow x = e - 1$.

Limits and Continuity

Formal Definition: The function *f(x)* has limit *L,* as *x* approaches *a*, denoted $\lim_{x\to a} f(x)=L$ if given any $\varepsilon>0$, there exists a $\delta>0$ such that |*f*(*x*) - *L*| < ε for all *x* satisfying 0 < |*x* – *a*|< δ.

PROPERTIES OF LIMITS:

$$\lim_{x\to a}[f(x)\pm g(x)]=\lim_{x\to a} f(x)\pm\lim_{x\to a} g(x),$$

$$\lim_{x\to a}[f(x)\cdot g(x)]=\lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x),$$

$$\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right]=\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)},$$

$$\lim_{x\to a}[f(x)]^n=\left[\lim_{x\to a} f(x)\right]^n$$

EXAMPLE:

Give an $\varepsilon-\delta$ proof of the addition property of limits: If $\lim_{x\to a} f(x)=L$ and $\lim_{x\to a} g(x)=K$, then $\lim_{x\to a}[f(x)+g(x)]=L+K$.

SOLUTION:

Let $\varepsilon>0$. Then $\frac{\varepsilon}{2}>0$.

Since $\lim_{x\to a} f(x)=L$, there exists $\delta_1>0$ such that if $|x-a|<\delta_1$, then $|f(x)-L|<\frac{\varepsilon}{2}$.

Also, since $\lim_{x\to a} g(x)=K$, there exists $\delta_2>0$, such that if $|x-a|<\delta_2$, then $|g(x)-K|<\frac{\varepsilon}{2}$.

Let $\delta=$ minimum(δ_1,δ_2).

Hence, if $|x-a|<\delta$, then $|x-a|<\delta_1$ and $|x-a|<\delta_2$, and therefore, $|f(x)-L|<\frac{\varepsilon}{2}$ and $|g(x)-K|<\frac{\varepsilon}{2}$.

So,

$$|[f(x)+g(x)]-(L+K)|=|[f(x)-L]+[g(x)-K]|$$

$\le|f(x)-L|+|g(x)-K|$ (by the triangle inequality) $<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$ as required.

Therefore, if $\lim_{x\to a} f(x)=L$ and $\lim_{x\to a} g(x)=K$, then $\lim_{x\to a}[f(x)+g(x)]=L+K$.

L'HOPITAL'S RULE:

If the limit of the quotient of differentiable functions $f(x)$ and $g(x)$ are of types $\frac{0}{0}$ or $\frac{\infty}{\infty}$, and if $g'(a)$ is not 0, then $\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$

EXAMPLE:

Evaluate $\lim_{x\to 2}\frac{x^3+3x^2-10x}{x^3-3x^2+2x}$.

SOLUTION:

Note first that this limit is in $\frac{0}{0}$ form. Therefore, use L'Hopital's theorem:

We find

$$\lim_{x\to 2}\frac{x^3+3x^2-10x}{x^3-3x^2+2x}=\lim_{x\to 2}\frac{\frac{d}{dx}(x^3+3x^2-10x)}{\frac{d}{dx}(x^3-3x^2+2x)}$$

$$=\lim_{x\to 2}\frac{3x^2+6x-10}{3x^2-6x+2}=\frac{14}{2}=7$$

TECHNIQUES FOR FINDING LIMITS:

- First try substitution, factoring
- If you're taking the limit as $x\longrightarrow\infty$ of a quotient, and substitution yields an invalid answer, then try first to divide both the numerator and the denominator of the limit expression by its highest power of *x*
- If the limit expression contains ***absolute values***, then try breaking the limit up into *two one-sided limits*.
- If the limit expression is a quotient, and if factoring doesn't work, then try l'Hopital's Rule

EXAMPLE:

Determine the following limit:

$$\lim_{x\to\infty}\frac{5x^3-4x+2}{3+x^2-6x^3}.$$

SOLUTION:

$$\lim_{x\to\infty}\frac{5x^3-4x+2}{3+x^2-6x^3}=\lim_{x\to\infty}\frac{x^3\left(5-\frac{4}{x^2}+\frac{2}{x^3}\right)}{x^3\left(-6+\frac{1}{x}+\frac{3}{x^3}\right)}$$

$$=\lim_{x\to\infty}\frac{5-0+0}{-6+0+0}=-\frac{5}{6}.$$

Continuity: a function is continuous at point *a* if and only if the left hand and right hand limits exist and left hand limit = right hand limit = *f(a)* (i.e. if $\lim_{x\to a} f(x)=f(a)$).

EXAMPLE:

If $f(x)=\begin{cases}x+e^K & if \quad x\le 1\\ \frac{2x^2-2x}{x-1} & if \quad x>1\end{cases}$ is continuous at *x* = 1, then determine the value of K.

SOLUTION:

We require: $\lim_{x\to 1-} f(x)=\lim_{x\to 1+} f(x)=f(1)$

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}(x+e^K)$$

$$=1+e^K=f(1)$$

$$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}\frac{2x^2-2x}{x-1}$$

$$=\lim_{x\to 1^+}\frac{(x-1)\cdot 2x}{x-1}=\lim_{x\to 1^+}2x=2$$

Therefore: 1 + e^K = 2 $\Longrightarrow$ *K* = 0.

Thus, if $f(x)$ is continuous at *x* = 1, then K = 0.

INTERMEDIATE VALUE THEOREM:

If $f(x)$ is continuous on $[a,b]$ and if *k* is between $f(a)$ and $f(b)$ then there exists a *c* such that $a<c<b$ and $f(c)=k$

EXAMPLE:

Use the Intermediate theorem to show that the equation *x* + sin *x* = 1 has a solution in the interval $\left[0,\frac{\pi}{2}\right]$.

SOLUTION:

Let *f*(*x*) = *x* + sin *x*. Then *f* is continuous on $\left[0,\frac{\pi}{2}\right]$.

f(0) = 0 + sin 0 = 0 and $f(\frac{\pi}{2})=\frac{\pi}{2}+\sin\frac{\pi}{2}=\frac{\pi}{2}+1$.

Thus, *f*(0) < 1 < $f(\frac{\pi}{2})$. By the I.V.T, there is a number *c* in the interval $\left(0,\frac{\pi}{2}\right)$ such that *f*(*c*) = 1, i.e. such that *c* is the solution to the expression *x* + sin *x* = 1.

SQUEEZE THEOREM:

If $f(x)\le g(x)\le h(x)$ near *a* and $\lim_{x\to a} f(x)=\lim_{x\to a} h(x)=L$ then $\lim_{x\to a} g(x)$ exists and is equal to L.

EXAMPLE:

If, for all values of *x* in the interval (0, 5), 1 + *x* ≤ *f* (*x*) ≤ 3 + sin π*x*, find $\lim_{x\to 2} f(x)$.

SOLUTION:

$\lim_{x\to 2} 1+x$ = 1 + 2 = 3

$\lim_{x\to 2} 3+\sin\pi x$ = 3 + sin 2π = 3.

⇒ $\lim_{x\to 2} f(x)$ = 3 by the squeeze theorem.