MATH 223 Study Sheet

PREP101 Your Competitive Edge

Systems of Linear Equations

INHOMOGENOUS: AX=Q OR:

$a_0x_0 + a_1x_1 + a_2x_2 = Q_1$
$b_0x_0 + b_1x_1 + b_2x_2 = Q_2$
$c_0x_0 + c_1x_1 + c_2x_2 = Q_3$

Rank= # leading 1's in row-reduced augmented matrix. If **rank=r** and **n=number of variables**, then **solution has n-r free parameters**.

1. **No solution, n – r < 0** (equations are inconsistent)
2. **Unique Solution, n - r = 0**
3. **Infinitely many solutions, n – r > 0**

HOMOGENEOUS: AX=0

(Right side is all zeros) Always has at least one solution **x=0.**
If **n - r > 0,** solution has n-r free parameters, otherwise x=0 is the only solution.

Matrices I (Matrix Basics)

MATRIX OPERATIONS:

Addition and subtraction:
$A \pm B = [a_{ij} \pm b_{ij}]$,

Scalar Multiplication:

$kA = [ka_{ij}]$

Matrix Multiplication:

Defined only if A is m x n, and B is n x p, then AB will be m x p. Find the (m,p)-entry as follows:

1. Take the mth row of A and pth column of B
2. Multiply corresponding entries
3. add the products.

So, $C = [c_{ij}] = AB$:

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Transpose:
If $A = [a_{ij}]$, then its transpose, $A^T = [a_{ji}]$
Properties:

1. $(A^T)^T = A$
2. $(kA)^T = kA^T$
3. $(A + B)^T = A^T + B^T$
4. $(AB)^T = B^TA^T$

if $A^T = A$, then it is called *Symmetric*
if $A^T = -A$ then it is called *Antisymmetric*

Properties of the Inverse:

1. $(A^{-1})A=A(A^{-1}) = I$ (definition of inverse)
2. $(A^{-1})^{-1} = A$
3. If A and B invertible, $(AB)^{-1} = B^{-1}A^{-1}$
4. $(A_1A_2\ldots A_k)^{-1} = A_k^{-1}\ldots A_2^{-1}A_1^{-1}$
5. $(A^k)^{-1} = (A^{-1})^k$
6. $(aA)^{-1} = 1/aA^{-1}$
7. $(A^T)^{-1} = (A^{-1})^T$

Vectors I (Vector Space Basics)

PROPERTIES OF A VECTOR SPACE:

1. u + v = v + u
2. u + (v + w) = (u + v) + w
3. 0 + u = u
4. u + (-u) = 0
5. k(u + v) = ku + kv
6. (k + p)u = ku + pu
7. (kp)u = k(pu)
8. 1u = u

Where k and p are scalars. Any set of objects obeying these properties is a vector space.

SUBSPACES:

A subset **U** of a vector space is a subspace if:

1. $0 \in U$
2. if $X, Y \in U$, then $X + Y \in U$
3. if $X \in U$, then $rX \in U$ for $r \in R$

If these hold, then all other properties of a vector space are automatically true for U.

LINEAR INDEPENDENCE, SPAN AND BASIS (DEFINITIONS)

Span of a set of vectors:
span($\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ **….**) is the set of all linear combinations
$c_1 \mathbf{v_1} + c_2 \mathbf{v_2} + \ldots$.
If span($\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ **….**) = U then we also say the vectors span U

Linear Independence
$v_1, v_2, v_3 \ldots$ are linearly independent if the only solution to
$c_1 v_1 + c_2 v_2 + c_3 v_3 + \ldots = 0$ is $c_1 = c_2 = c_3 = \ldots = 0$.

Linearly dependent = not linearly independent. Linear dependence means at least one of the vectors can be written as a linear combination (or sum of multiples) of some of the others.

Basis (of a vector space U) =
a linearly independent set of vectors that span U.

Dimension =
number of vectors in any basis of U (All bases for a given space have the same size, and any set that spans U contains at least dim U vectors.)

VECTOR SPACES RELATED TO AN M X N MATRIX A:

row space = the span of the rows of A = rowA

Column space = span of columns of A = cola

Null space or **kernel** = null A = ker A= $\{X \in R^n | AX = 0\}$

Image = im A = $\{Y \in R^m | Y = AX$ for some $X \in R^n\}$

Column space and image are the same:
col A = im A

RANK THEOREM

dim(rowA) = dim(colA) = rankA
Also, if A is m X n with rank r, then:

1. rankA = $\text{rank}A^T$
2. if A: m x n, rankA $\leq$ min(m,n)
3. dim(null A) =n-r
4. an n X n matrix A is invertible if and only if rankA = n.

HOW TO FIND A BASIS FOR THE ROW SPACE:

Row reduce the matrix. The nonzero rows form a basis for row A.

HOW TO FIND A BASIS FOR THE COLUMN SPACE:

Row reduce. The columns of the original (unreduced) matrix in the columns corresponding to the ones where the leading 1's occur for a basis for col A.

How To Find A Basis For The Null Space:

Null A is the solution set for the homogeneous equations AX=0. Therefore, solve AX=0 by row reducing the matrix, and write the solution in the form s $\mathbf{v_1}$ + t $\mathbf{v_2}$ + u $\mathbf{v_3}$ + … where $\mathbf{v_1}$, $\mathbf{v_2}$, $\mathbf{v_3}$… are column vectors. Then those vectors form a basis for null A.

A Theorem About Invertibility:

If A is an n X n matrix, the following are equivalent:

1. A is invertible
2. The columns of A are linearly independent
3. The columns of A span R^n
4. The rows of A are linearly independent
5. The rows of A span R^n
6. imA = R^n
7. ker A = 0.

Matrices II (Determinants)

Definition of Cofactors and Determinant

A: n x n matrix. Let A_{ij} be the(n-1) x (n-1) matrix obtained from A by deleting the row i and the column j. The (i,j) -cofactor of A is defined to be $C_{ij} = (-1)^{i+j}\det(A_{ij})$ and $\det(A) = a_{i1}C_{i1} + \ldots + a_{in}C_{in}$ (This can be used recursively to calculate a determinant.)

Properties of Determinants

1. det(AB) = det A det B
2. A is invertible ⇔ det A ≠ 0. In this case,
3. $\det\left(A^{-1}\right) = (\det A)^{-1} = \dfrac{1}{\det A}$.
4. $\det(A^T) = \det A$
5. $\det(kA) = k^n \det A$
6. If any row of A is 0, detA=0.
7. If A is triangular, detA is the product of elements along the main diagonal.

Determinants and Elementary Row Operations:

1. Multiplying one row by k multiplies detA by k.
2. Adding two rows doesn't change the determinant
3. Interchanging two rows multiplies the determinant by -1.

Cramer's Rule

Consider the inhomogeneous equations AX = B.
If A is invertible, then $X = A^{-1}B$ and

$$x_i = \frac{1}{\det A}\left(b_1C_{1i} + b_2C_{2i} + \cdots + b_nC_{ni}\right) = \frac{\det A_i}{\det A},$$

Where A_i is the matrix obtained by replacing column i in A by B.

Classical Adjoint

The classical adjoint of A, adj(A), is the transpose of the cofactor matrix: $\operatorname{adj}(A) = [C_{ij}]^T$

A(adj(A)) = (adj (A)) A = (det A) I.

If det A ≠ 0, then $A^{-1} = \dfrac{1}{\det A}$ adj (A).

Vectors II (Eigenvectors, Orthogonality, Linear Transformations)

Eigenvectors, Eigenvalues and Diagonalization

For an n X n matrix A:

• λ is an eigenvalue of A if there is a nonzero X s.t.

$$AX = \lambda X$$

And X is called the eigenvector corresponding to that eigenvalue.

Steps to Diagonalize a n x n Matrix:

1. **Find the eigenvalues** of A by solving the characteristic equation det(λI – A)=0
2. For each eigenvalue, calculate the basic solutions of (λI – A)X = 0 : a basis for null(λI – A). **These are the eigenvectors**. If there are n basic solutions in total, then A is diagonalizable.
3. Construct P whose columns are the basic solutions
4. $P^{-1}AP$ is the diagonal matrix.

Dot Product, Orthogonality

Given $X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $Y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$ in R^n, the dot product of X and Y is defined by
$X \cdot Y = X^TY = x_1y_1 + x_2y_2 + \ldots + x_ny_n$.
The length $||X||$ of X is defined by

$$\|X\| = \sqrt{X \cdot X} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

1. Two vectors X and Y are orthogonal if X·Y = 0.
2. A set $\{X_1, X_2, \ldots, X_m\}$ of nonzero vectors in R^n is called an orthogonal set if $X_i \cdot X_j = 0$ for i ≠ j.
3. An orthogonal set $\{X_1, X_2, \ldots, X_m\}$ is orthonormal if $\|X_i\| = 1$ for all i.

Every orthogonal set of vectors in R^n is linearly independent.

Projections, Expansion Coefficients

Expanding a Vector in a Basis

If $\{X_1, X_2, \ldots, X_n\}$ is an orthogonal basis of R^n, then

$$X = \frac{X \cdot X_1}{\|X_1\|^2} X_1 + \frac{X \cdot X_2}{\|X_2\|^2} X_2 \cdots + \frac{X \cdot X_n}{\|X_n\|^2} X_n$$

for every X in R^n.

Projections

Let $\{X_1, X_2 \ldots, X_m\}$ be an orthogonal basis of a subspace U of R^n. Given X in R^n, define

$$proj_U(X) = \frac{X \cdot X_1}{\|X_1\|^2} X_1 + \frac{X \cdot X_2}{\|X_2\|^2} X_2 + \cdots + \frac{X \cdot X_m}{\|X_m\|^2} X_m$$

Gram Schmidt Orthogonalization

Given any basis B = {Y1, … , Ym} of U, construct an orthogonal basis X1, … , Xm in U as follows:

$$X_1 = Y_1$$
$$X_2 = Y_2 - \frac{Y_2 \cdot X_1}{\| X_1 \|^2} X_1$$
$$X_3 = Y_3 - \frac{Y_3 \cdot X_2}{\| X_1 \|^2} X_1 - \frac{Y_3 \cdot X_2}{\| X_2 \|^2} X_2$$
$$\vdots$$
$$X_m = Y_m - \frac{Y_m \cdot X_1}{\| X_1 \|^2} X_1 - \frac{Y_m \cdot X_2}{\| X_2 \|^2} X_2 - \ldots - \frac{Y_m \cdot X_{m-1}}{\| X_{m-1} \|^2} X_{m-1}$$